Nonlinear Perturbation Theory for the Large Scale Structure

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Lecture 3

Outline

- brief review of statistical field theory
- the setup: Eulerian vs Lagrangian, equations of motion
- structure formation in the LineLand (1+1 dimensions)
- Standard Perturbation Theory
- performance and problems of SPT (response functions)
- IR effects: resummations and BAO’s
- UV behavior: Effective approaches

(drop the time dependence)

$$\frac{\partial}{\partial \tau}\delta_R(\mathbf{x}) + \frac{\partial}{\partial x^i}\left[(1+\delta_R(\mathbf{x}))v_R^i(\mathbf{x})\right] = 0$$

continuity eq.

$$\frac{\partial}{\partial \tau}v_R^i(\mathbf{x}) + \mathcal{H}v_R^i(\mathbf{x}) + v_R^k(\mathbf{x})\frac{\partial}{\partial x^k}v_R^i(\mathbf{x}) = -\nabla_x^i\phi_R(\mathbf{x}) - J_\sigma^i(\mathbf{x}) - J_1^i(\mathbf{x})$$

Euler eq.

short-distance effects

$$J_\sigma^i(\mathbf{x}) \equiv \frac{1}{1+\delta_R(\mathbf{x})}\frac{\partial}{\partial x^k}\left((1+\delta_R(\mathbf{x}))\sigma_R^{ki}(\mathbf{x})\right)$$

$$J_1^i(\mathbf{x}) \equiv \frac{1}{1+\delta(\mathbf{x})}\left(\langle(1+\delta)\nabla^i\phi\rangle_R(\mathbf{x}) - (1+\delta_R)(\mathbf{x})\nabla^i\phi_R(\mathbf{x})\right)$$

To close the system, we must provide information on the short-distance effects

Buchert, Dominguez, ’05, Pueblas Scoccimarro, ’09, Baumann et al. ’10

M.P., G. Mangano, N. Saviano, M. Viel, 1108.5203, Carrasco, Hertzberg, Senatore,1206.2976 ....

Single stream approximation

Set $\sigma^{ij} = \omega^{ijk} = \cdots = \nabla \times \mathbf{v} = 0$

...+ no higher moments, no vorticity,...

$$f(\mathbf{x}, \mathbf{p}, \tau) = \bar{\rho}\,(1 + \delta(\mathbf{x}, \tau))\,\delta_D(\mathbf{p} - am\nabla\varphi(\mathbf{x}, \tau))$$

System described by $\delta(\mathbf{x}, \tau)\,, \quad \theta(\mathbf{x}, \tau) \equiv \nabla^2\varphi(\mathbf{x}, \tau) \equiv \nabla \cdot \mathbf{v}(\mathbf{x}, \tau)$

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot ((1 + \delta)\mathbf{v})$$ continuity

$$\frac{\partial\, \mathbf{v}}{\partial\, \tau} + \mathcal{H}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\phi$$ Euler

$$\nabla^2\phi = \frac{3}{2}\,\Omega_M\,\mathcal{H}^2\,\delta$$ Poisson

warning: self-consistent … but ultimately wrong!

Linear order solution

$$\frac{\partial \delta}{\partial \tau} + \theta = 0$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta = -\nabla^2 \phi$$

$$\nabla^2 \phi = \frac{3}{2}\,\Omega_M\,\mathcal{H}^2\,\delta$$

$$\Longrightarrow \quad \ddot{\delta} + \mathcal{H}\dot{\delta} = \frac{3}{2}\Omega_M \mathcal{H}^2 \delta$$

linear GR equation for $k \gg \mathcal{H}$

Solution : $\delta^{(1)}(\mathbf{k},\tau) = -\dfrac{\theta^{(1)}(\mathbf{k},\tau)}{\mathcal{H} f_\pm} = \delta(\mathbf{k},\tau_{in}) D_\pm(\tau)$ $\quad (D_\pm(\tau_{in}) = 1)$ $\quad \left(f_\pm \equiv \dfrac{d \ln D_\pm}{d \ln a} \right)$

growing / decaying mode

For EdS (Ω_M=1): $\quad D_\pm = \left(\dfrac{a(\tau)}{a(\tau_{in})} \right)^{f_\pm} \qquad f_+ = 1,\; f_- = -3/2$

Compact notation

$$\varphi_a(\mathbf{k},\eta) = e^{-\eta}\left(\begin{array}{c}\delta(\mathbf{k},\eta)\\ -\theta(\mathbf{k},\eta)/(\mathcal{H}f)\end{array}\right)$$

$$\eta \equiv \log D(\tau)/D(\tau_0)$$

The continuity+Euler+Poisson system reads:

$$\left(\delta_{ab}\partial_\eta + \Omega_{ab}(\eta)\right)\varphi_b(\mathbf{k},\eta) = e^{\eta}\int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\gamma_{abc}(\mathbf{k},\mathbf{q},\mathbf{p})\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)$$

linear

$$\Omega_{ab}(\eta) = \left(\begin{array}{cc} 1 & -1\\ -\frac{3}{2}\frac{\Omega_m(\eta)}{f^2(\eta)} & \frac{3}{2}\frac{\Omega_m(\eta)}{f^2(\eta)}\end{array}\right)$$

nonlinear

$$\gamma_{112}(\mathbf{k},\mathbf{q},\mathbf{p}) = \gamma_{121}(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{\mathbf{k}\cdot\mathbf{p}}{2p^2}$$

$$\gamma_{222}(\mathbf{k},\mathbf{q},\mathbf{p}) = \frac{k^2\,\mathbf{q}\cdot\mathbf{p}}{2\,q^2p^2}$$

SPT=Iterative solution

$$\varphi_1^{(1)}(\mathbf{k},\eta) = \varphi_2^{(1)}(\mathbf{k},\eta) \equiv \varphi^{(1)}(\mathbf{k})$$

linear solution

$$g_{ab}(\eta) = \left[\begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix} + e^{-5/2\,\eta}\begin{pmatrix} 2/5 & -2/5 \\ -3/5 & 3/5 \end{pmatrix}\right]\Theta(\eta)$$

linear propagator

$$\varphi_a^{(2)}(\mathbf{k},\eta) = \int_0^\eta ds\, g_{ab}(\eta - s)\, e^s\, I_{\mathbf{k},\mathbf{q},\mathbf{p}}\gamma_{bcd}(\mathbf{k},\mathbf{q},\mathbf{p})\varphi_c^{(1)}(\mathbf{q},s)\varphi_d^{(1)}(\mathbf{p},s)$$

$$= e^{2\eta} I_{\mathbf{k},\mathbf{q_1},\mathbf{q_2}}\, F_a^{(2)}(\mathbf{q}_1,\mathbf{q}_2)\varphi^{(1)}(\mathbf{q}_1)\varphi^{(1)}(\mathbf{q}_2)$$

2nd order solution

$$\left(I_{\mathbf{k},\mathbf{q_1},\cdots,\mathbf{q_n}} \equiv \int \frac{d^3q_1}{(2\pi)^3}\cdots\frac{d^3q_n}{(2\pi)^3}\delta_D(\mathbf{k} - \sum_{i=1}^n \mathbf{q_i})\right)$$

$\cdots$

$$\varphi_a^{(n)}(\mathbf{k},\eta) = e^{n\eta} I_{\mathbf{k},\mathbf{q_1},\cdots,\mathbf{q}_n}\, F_a^{(n)}(\mathbf{q}_1,\cdots,\mathbf{q}_n)\varphi^{(1)}(\mathbf{q}_1)\cdots\varphi^{(1)}(\mathbf{q}_n)$$

nth order solution

If the initial conditions are gaussian, then only correlators involving an even number of initial fields are non-vanishing

tree-level

Power spectrum: $$\langle\varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{k}',\eta)\rangle = \langle\varphi_a^{(1)}(\mathbf{k},\eta)\varphi_b^{(1)}(\mathbf{k}',\eta)\rangle$$

one-loop

$$+\langle\varphi_a^{(1)}(\mathbf{k},\eta)\varphi_b^{(3)}(\mathbf{k}',\eta)\rangle + \langle\varphi_a^{(3)}(\mathbf{k},\eta)\varphi_b^{(1)}(\mathbf{k}',\eta)\rangle$$

$$+\langle\varphi_a^{(2)}(\mathbf{k},\eta)\varphi_b^{(2)}(\mathbf{k}',\eta)\rangle + O((\varphi^{in})^6)$$

$+ \ldots$

Bispectrum: $$\langle\varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{k}',\eta)\varphi_c(\mathbf{k}'',\eta)\rangle = \langle\varphi_a^{(2)}(\mathbf{k},\eta)\varphi_b^{(1)}(\mathbf{k}',\eta)\varphi_c^{(1)}(\mathbf{k}'',\eta)\rangle$$

tree-level

$$+2 \text{ permutations} + O((\varphi^{in})^6)$$

Computation of the Power Spectrum

$$P(k,\tau) = D^2(\tau)P_{11}(k) + D^4(\tau)\left[P_{13}(k) + P_{22}(k)\right] + \dots$$

linear PS: output from CAMB, CLASS, …

$$P_{13}(k) = \frac{k^3 P_{11}(k)}{252\,(2\pi)^2}\int_0^\infty dr P_{11}(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}\left(r^2-1\right)^3\left(7r^2+2\right)\ln\left|\frac{1+r}{1-r}\right|\right]$$

$$P_{22}(k) = \frac{k^3}{98\,(2\pi)^2}\int_0^\infty dr P_{11}(kr)\int_{-1}^{1} dx P_{11}\left[k\left(1+r^2-2rx\right)^{1/2}\right]\frac{\left(3r+7x-10rx^2\right)^2}{\left(1+r^2-2rx\right)^2}$$

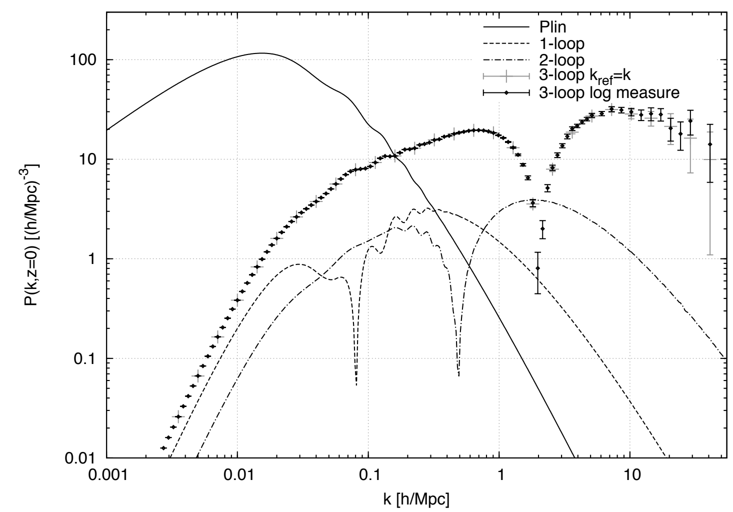
Cosmology information in $P_{11}(k)$ **and** $D(\tau)$

Integrals to be performed numerically for ΛCDM…

FFTLog approach

Simonovic et al 1708.08130

Fourier Transform the PS (with respect to log k)



Log k

$$\bar{P}_{\rm lin}(k_n) = \sum_{m=-N/2}^{m=N/2} c_m \, k_n^{\nu + i\eta_m}$$

(ν is a parameter)

$$\eta_m = \frac{2\pi m}{\log(k_{\rm max}/k_{\rm min})} \qquad c_m = \frac{1}{N}\sum_{l=0}^{N-1} P_{\rm lin}(k_l)\, k_l^{-\nu} k_{\rm min}^{-i\eta_m}\, e^{-2\pi i m l/N}$$

$$P_{22}(k) = 2\int_{\boldsymbol{q}} F_2^2(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) P_{\rm lin}(q) P_{\rm lin}(|\boldsymbol{k}-\boldsymbol{q}|)$$

$$F_2(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) = \frac{5}{14} + \frac{3k^2}{28q^2} + \frac{3k^2}{28|\boldsymbol{k}-\boldsymbol{q}|^2} - \frac{5q^2}{28|\boldsymbol{k}-\boldsymbol{q}|^2} - \frac{5|\boldsymbol{k}-\boldsymbol{q}|^2}{28q^2} + \frac{k^4}{14|\boldsymbol{k}-\boldsymbol{q}|^2 q^2}$$

The 1-loop integral becomes a combination of

$$\int_{\boldsymbol{q}} \frac{1}{q^{2\nu_1}|\boldsymbol{k}-\boldsymbol{q}|^{2\nu_2}} \equiv k^{3-2\nu_{12}} \mathsf{I}(\nu_1,\nu_2)$$

with $$\mathsf{I}(\nu_1,\nu_2) = \frac{1}{8\pi^{3/2}} \frac{\Gamma(\frac{3}{2}-\nu_1)\Gamma(\frac{3}{2}-\nu_2)\Gamma(\nu_{12}-\frac{3}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(3-\nu_{12})}$$

(+ symetries and recursion relations …)

$$\bar{P}_{22}(k) = k^3 \sum_{m_1,m_2} c_{m_1} k^{-2\nu_1} \cdot M_{22}(\nu_1,\nu_2) \cdot c_{m_2} k^{-2\nu_2}$$

Cosmology dependence (PS shape)

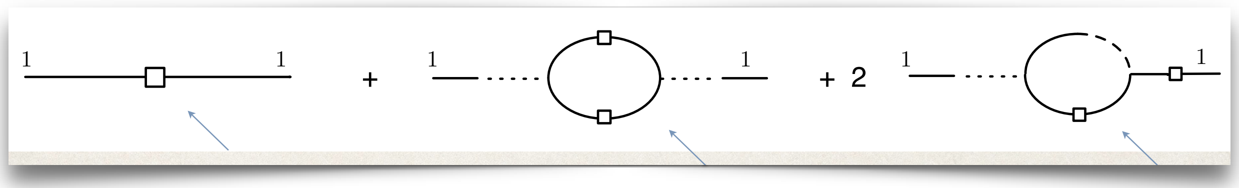
Integrals done once forever

$$M_{22}(\nu_1,\nu_2) = \frac{(\frac{3}{2}-\nu_{12})(\frac{1}{2}-\nu_{12})[\nu_1\nu_2(98\nu_{12}^2-14\nu_{12}+36)-91\nu_{12}^2+3\nu_{12}+58]}{196\,\nu_1(1+\nu_1)(\frac{1}{2}-\nu_1)\,\nu_2(1+\nu_2)(\frac{1}{2}-\nu_2)} \mathsf{I}(\nu_1,\nu_2)$$

Loop integral —> FFT+matrix multiplication: Very Fast!!

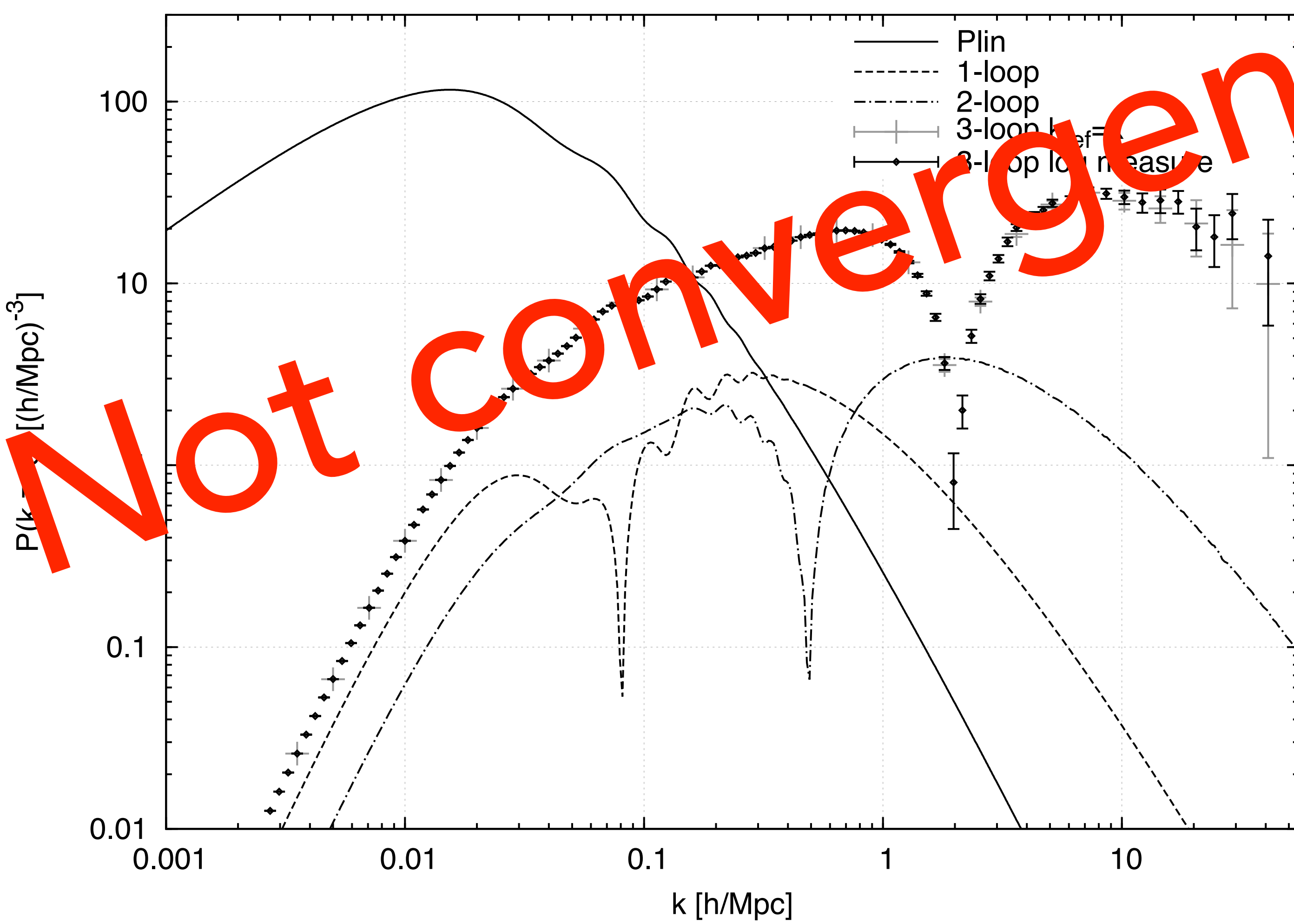
Performance of Standard PT

$$P(k,z) = D(z)^2 P^{(1)}(k) + D(z)^4 F^{(1l)}(k) + D(z)^6 F^{(2l)}(k) + \cdots$$

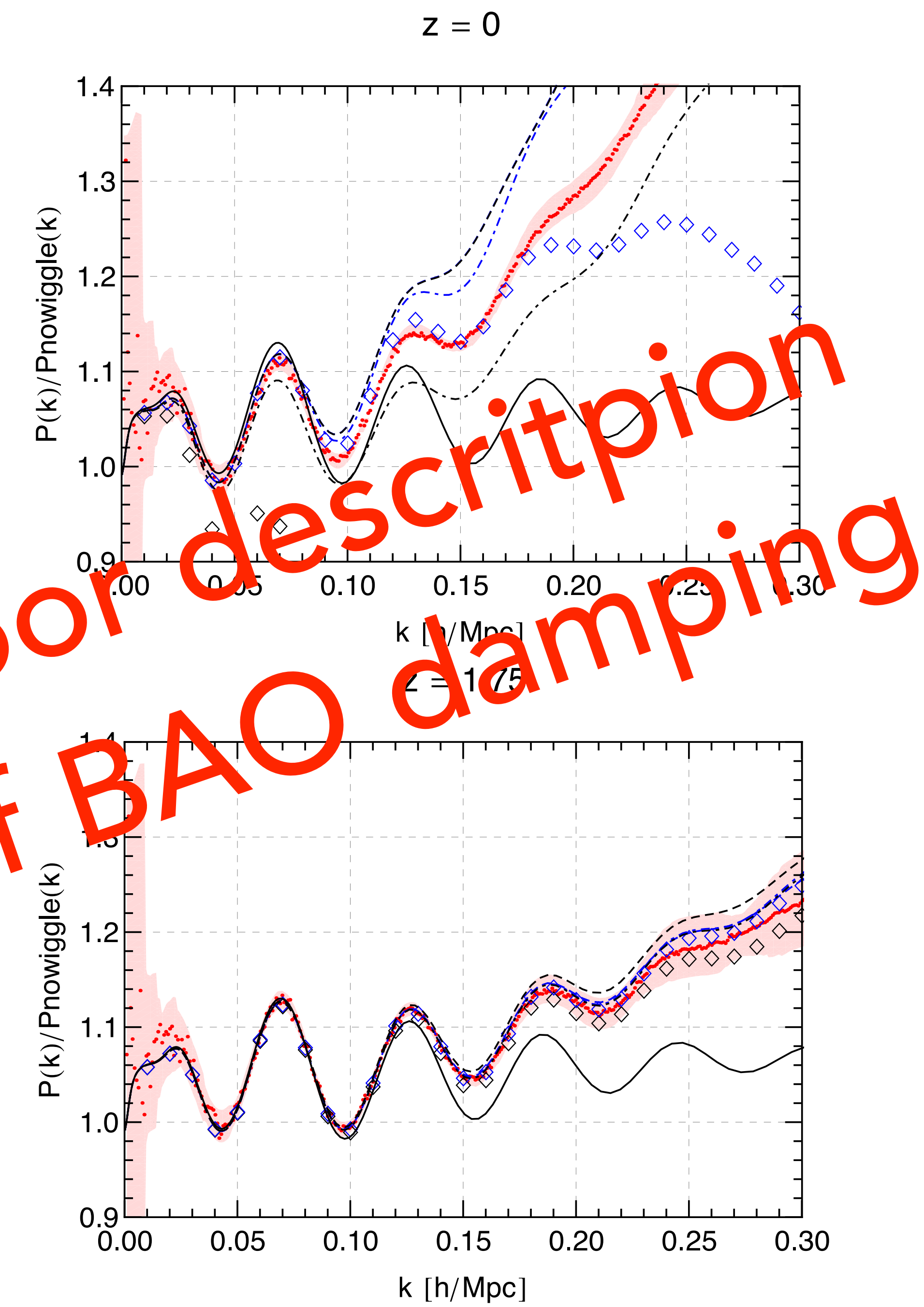


linear

1-loop



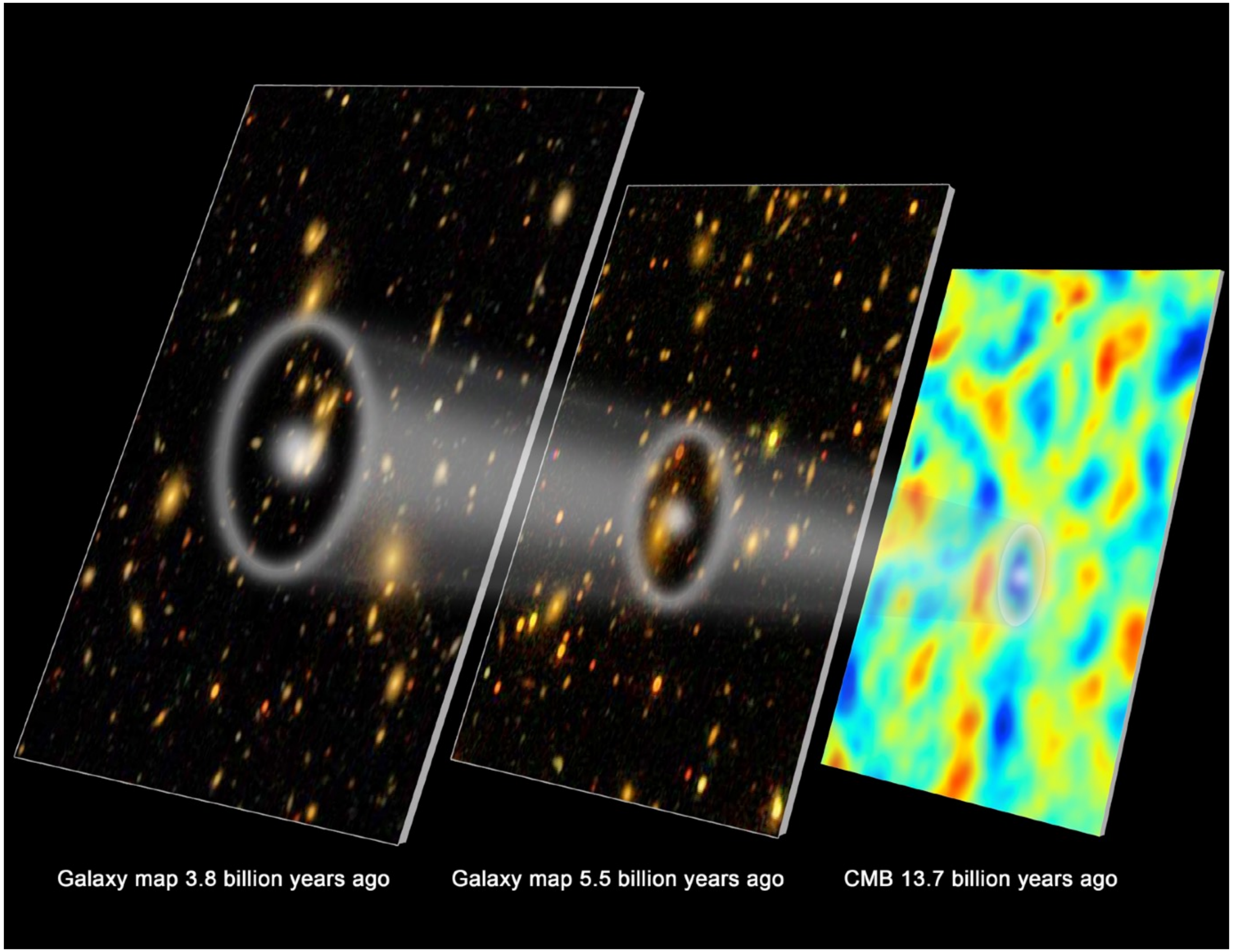
Not convergent !



poor descritpion of BAO damping

Blas et al. 1309.3308

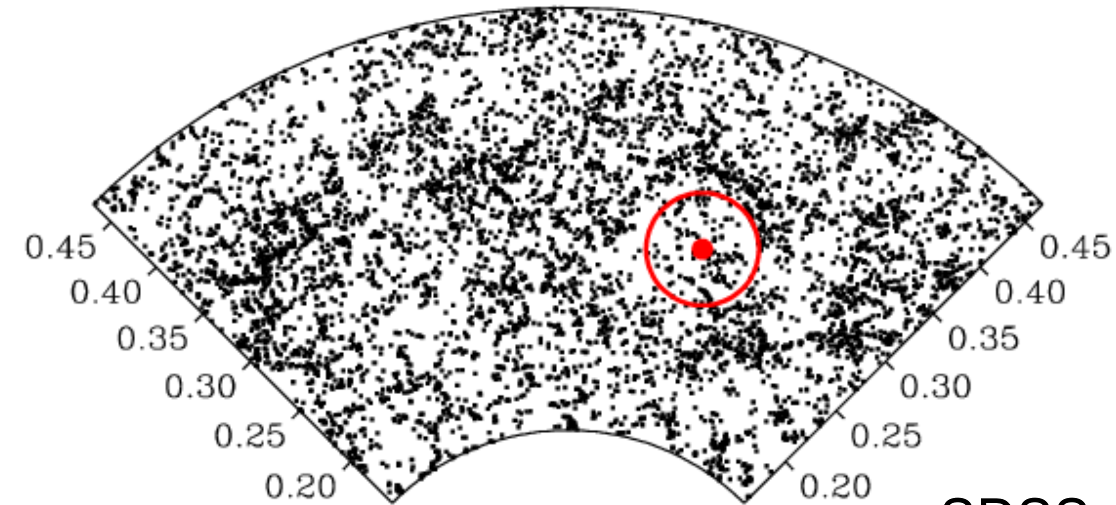
Baryonic Acoustic Osicllations



Credit: Eric Huff, the SDSS-III team, and the South Pole Telescope team

See Vanina's (and Marco's) lectures!

$$s = \frac{1}{H_0 \Omega_m^{1/2}} \int_0^{a_*} da \frac{c_s}{(a + a_{eq})^{1/2}} \simeq 110 \mathrm{Mpc}\, h^{-1}$$



SDSS

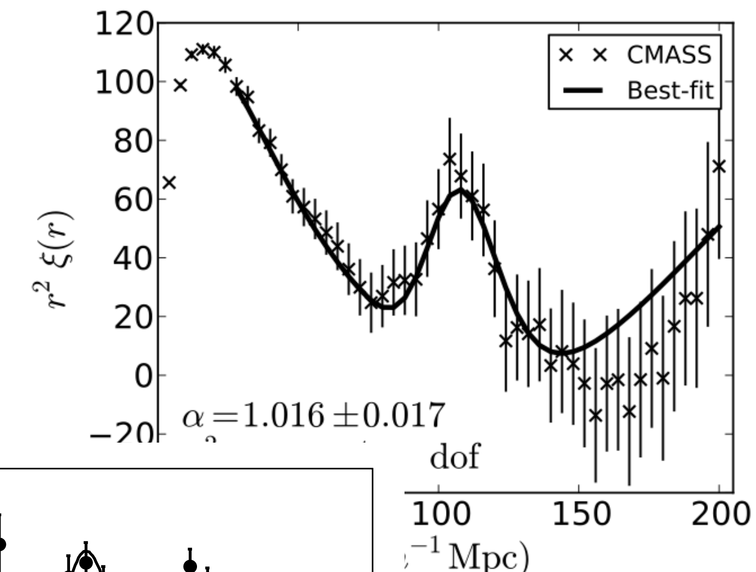
comoving sound horizon at recombination

same comoving scale seen in CMB anisotropies

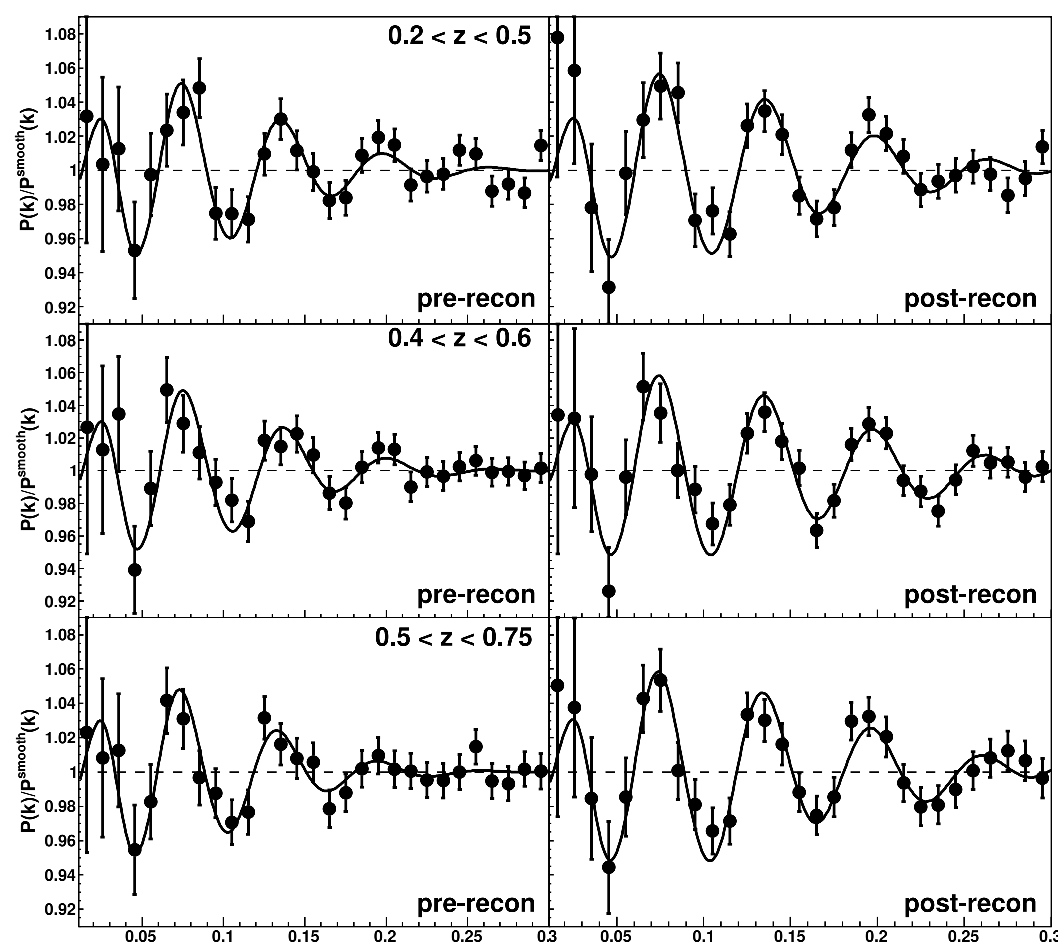
and in LSS at different redshifts.

Map from comoving to (angular and diameter) distances

is cosmology-dependent: STANDARD RULER

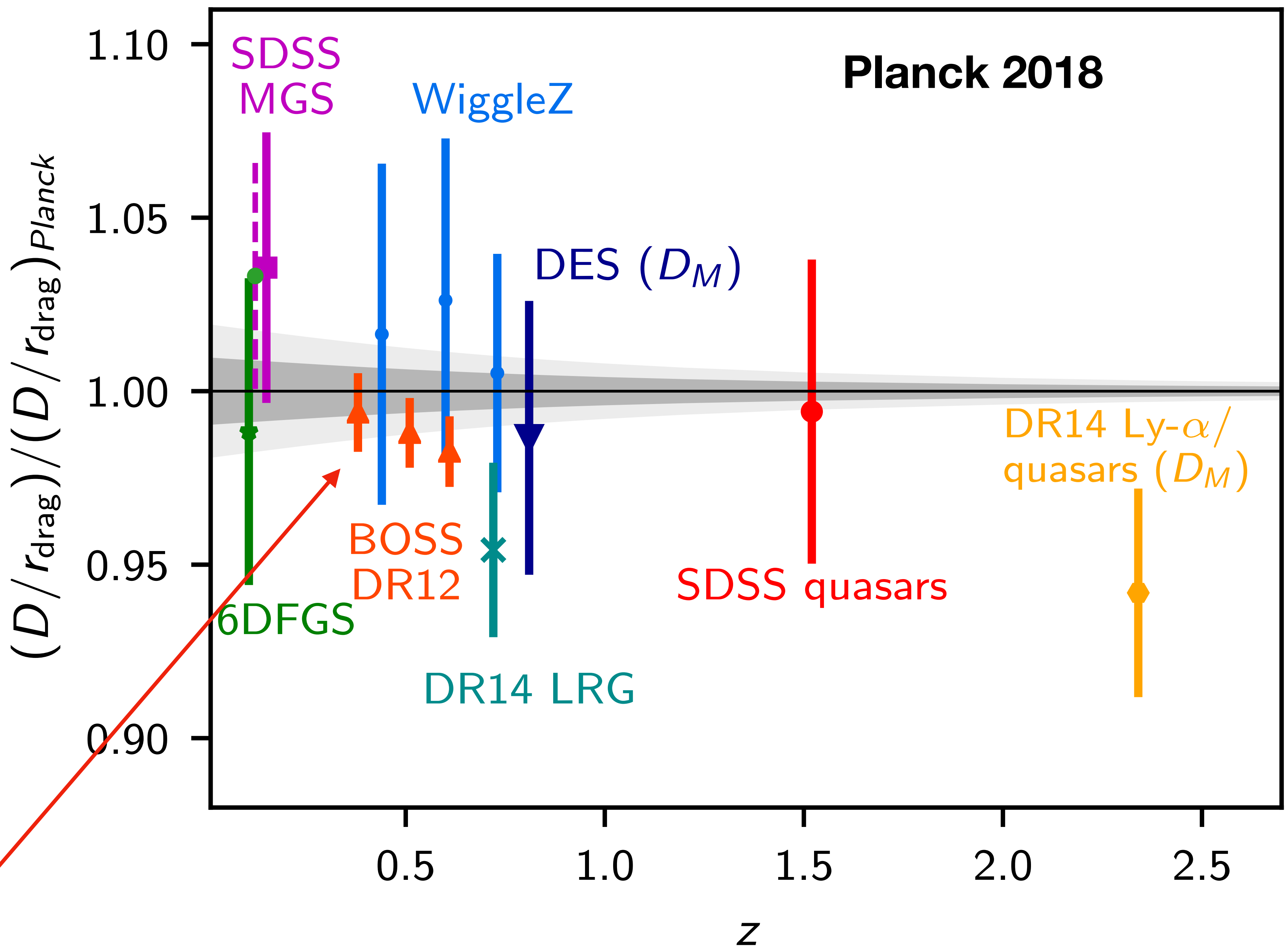


$$k_{BAO} = \frac{2\pi}{s} \simeq 0.057\; h/\mathrm{Mpc}$$



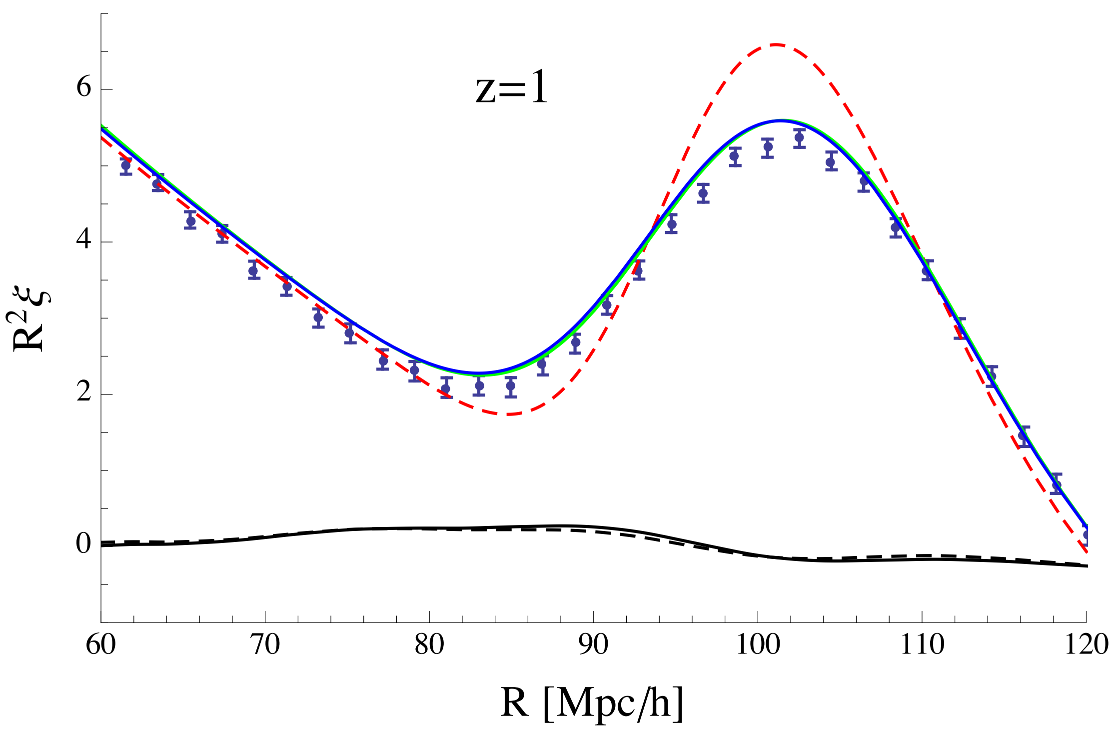
Beutler et al '16

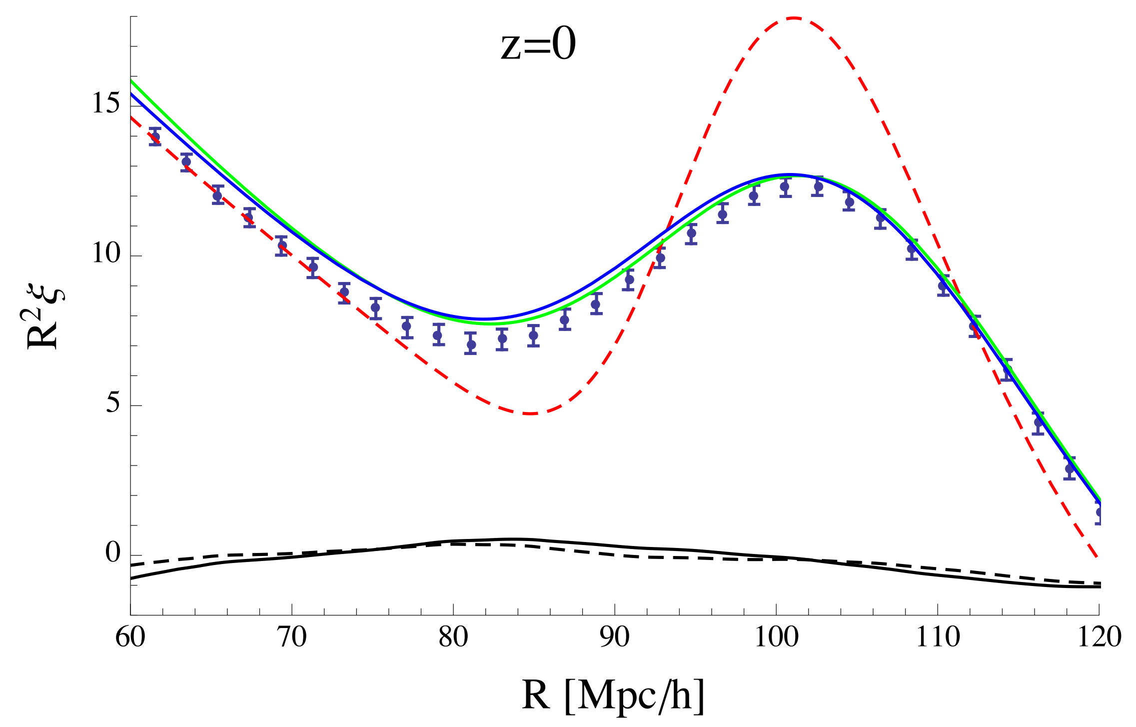
sound horizon from LSS vs. Planck

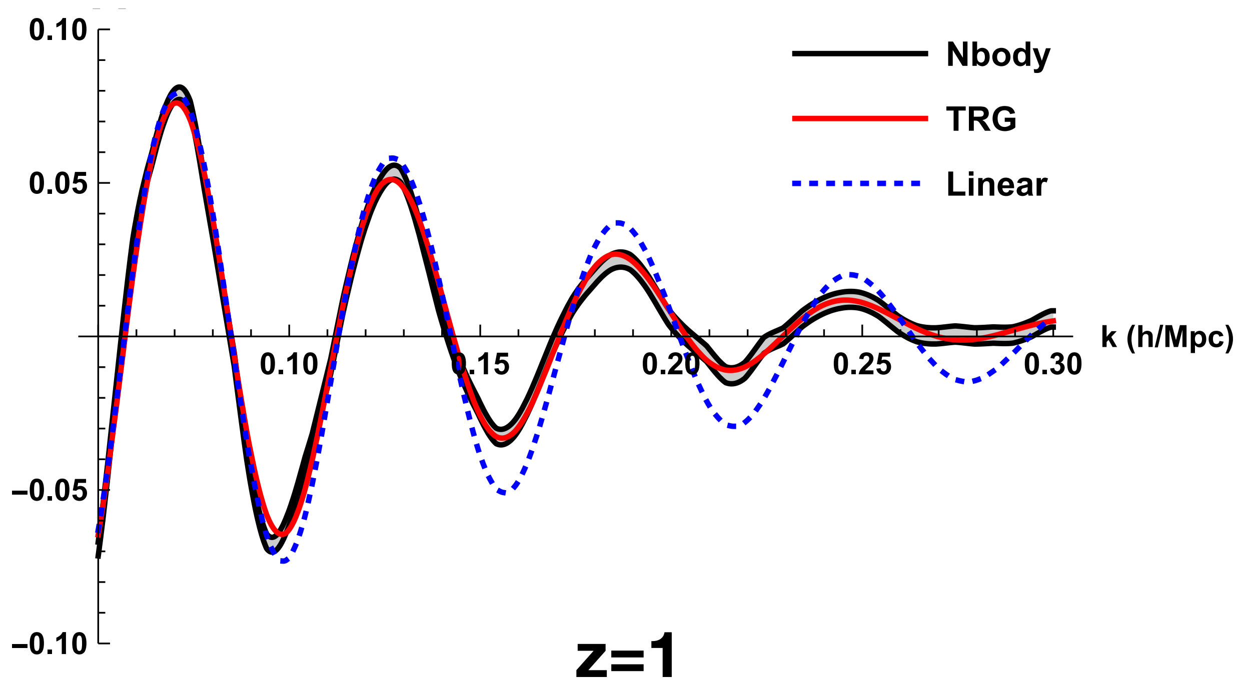


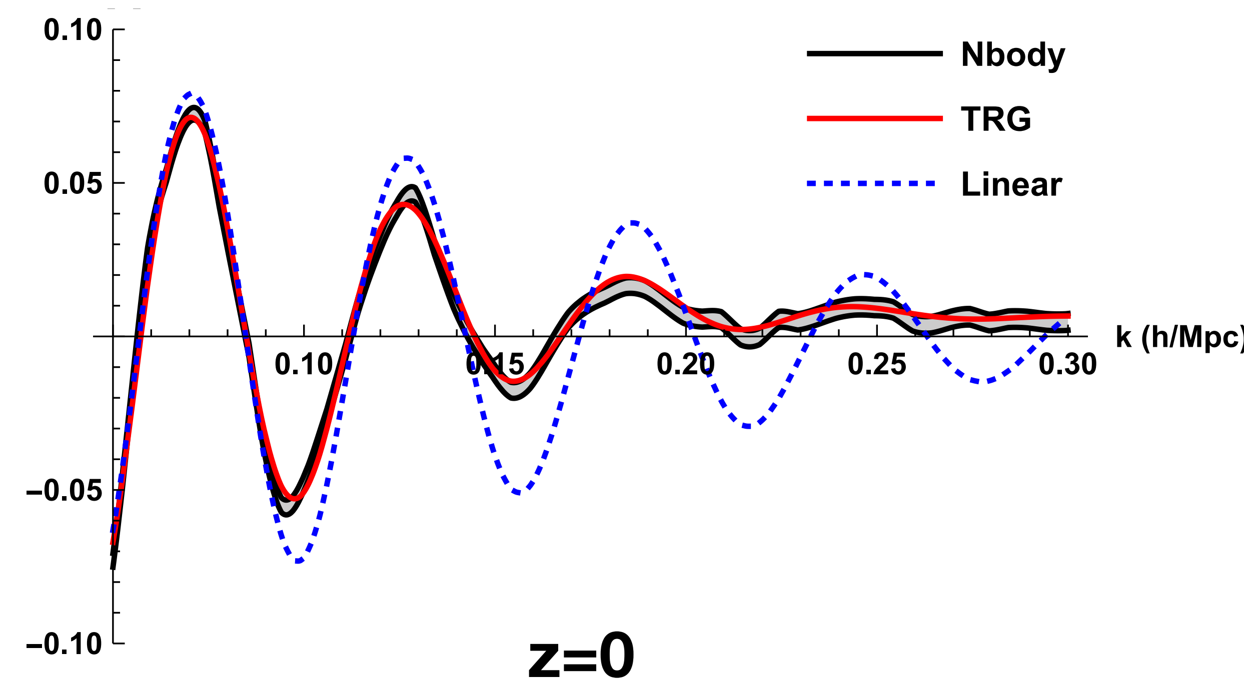
< 2 % error!!

nonlinear effects on BAO's

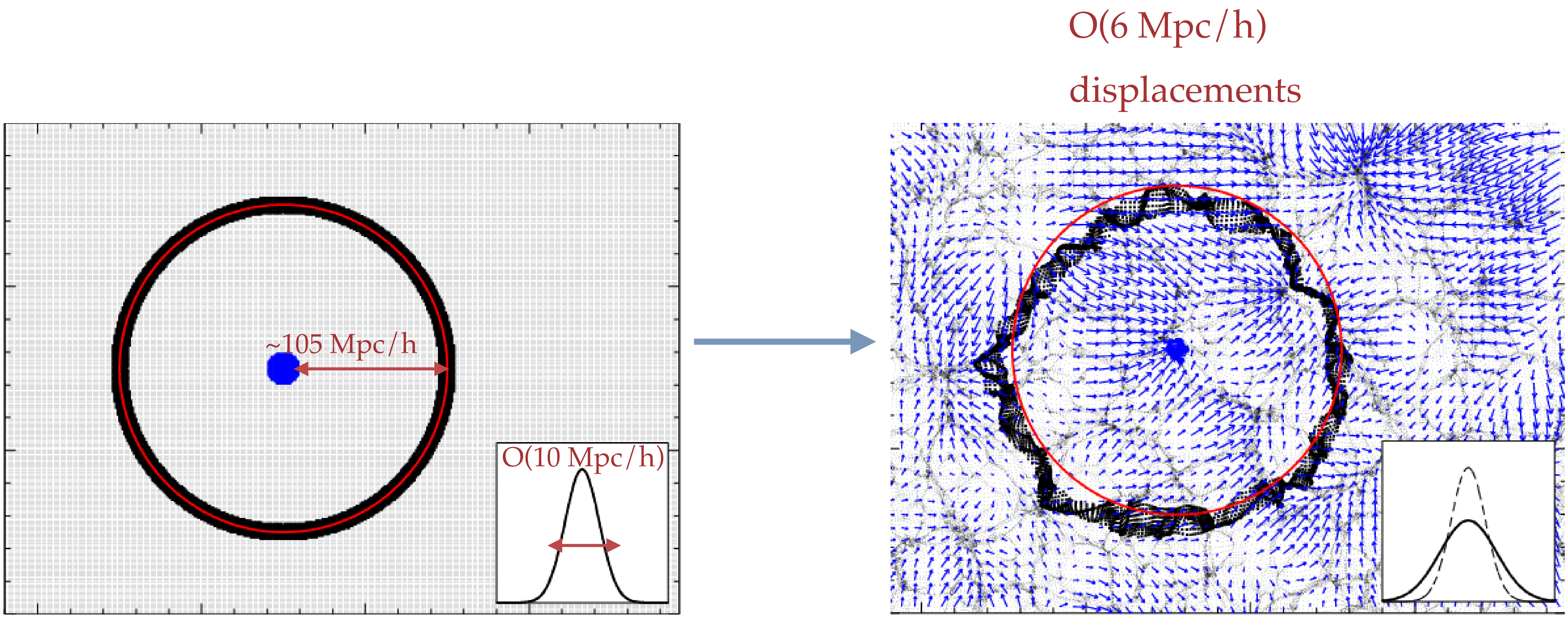








random displacements



Padmanabhan et al 1202.0090

$$\langle \psi^2(0) \rangle = \int \frac{d^3p}{(2\pi)^3} \langle \tilde{\psi}(\mathbf{p})^2 \rangle' = \int \frac{d^3p}{(2\pi)^3} \frac{P(p)}{p^2} \simeq (6\,\mathrm{Mpc/h})^2$$

(in linear theory)

how to include random displacements (sketch)

effect of a long(~ 10-100 Mpc/h) wavelength displacement on a short (< 10 Mpc/h) wavelength density perturbation

$$\delta(\mathbf{x}) \simeq \int d^3q\, \delta_{\rm lin}(\mathbf{q})\, \delta_D(\mathbf{q} + \psi(\mathbf{q}) - \mathbf{x})$$

$$\delta(\mathbf{k}) \simeq \int d^3q\, \delta_{\rm lin}(\mathbf{q})\, e^{i\mathbf{k}\cdot(\mathbf{q}+\psi(\mathbf{q}))}$$

$$P(k) = \langle \delta(\mathbf{k})\delta(-\mathbf{k})\rangle' \simeq \int d^3r\, \xi_{\rm lin}(r)\, e^{i\mathbf{k}\cdot\mathbf{r}} \langle e^{i\mathbf{k}\cdot\Delta\psi(r)}\rangle$$

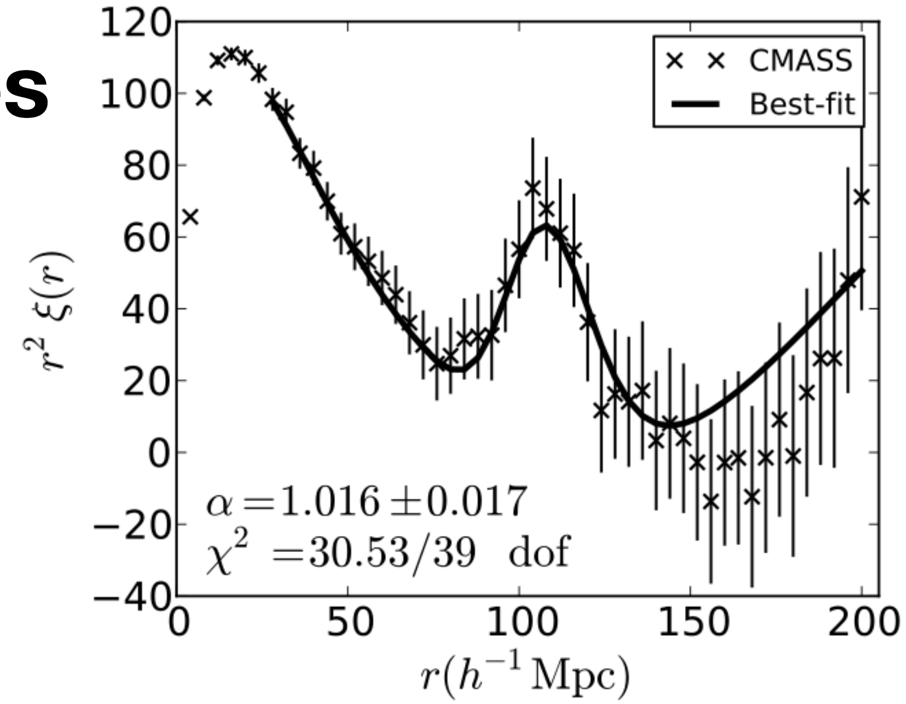
gaussian field

$$\simeq e^{-\frac{k^2\langle\Delta\psi(r)^2\rangle}{2}}$$

Exponential damping!

$$\Delta\psi(r) \equiv \psi(r/2) - \psi(-r/2)$$

Effect on the BAO peak/wiggles



$$\xi_{\rm lin}(r) = \xi_{\rm lin,smooth}(r) + \xi_{\rm lin,peak}(r)$$

$$P_{\rm wiggle}(k) = \int d^3r\, \xi_{\rm lin,peak}(r)\, e^{i\mathbf{k}\cdot\mathbf{r}} e^{-\frac{k^2\langle\Delta\psi(r)^2\rangle}{2}} \simeq e^{-\frac{k^2\langle\Delta\psi(r_s)^2\rangle}{2}} P_{\rm linear,wiggle}(k)$$

$$\langle\Delta\psi(r_s)^2\rangle = \frac{1}{6\pi^2}\int dp\, P_{\rm lin}(p)\,(1 - j_0(p\,r_s) + 2j_2(p\,r_s))$$

Different ways to include displacements lead to equivalent results.
Well understood theoretically.
Not a nuisance, but a *calculable physical effect*!!

BOSS analysis (Beutler et al '16)

$$P(k) = B^2 P_{\mathrm{sm,lin}}(k) F_{\mathrm{fog}}(k, \Sigma_s) \left[1 + \left(\frac{P_{\mathrm{lin}}(k)}{P_{\mathrm{sm,lin}}(k)} - 1\right) \mathrm{e}^{-k^2 \Sigma_{\mathrm{nl}}^2/2}\right]$$

$$\frac{a_{0,1}}{k^3} + \frac{a_{0,2}}{k^2} + \frac{a_{0,3}}{k} + a_{0,4} + a_{0,5}\, k$$

nuisance parameters

Noda, MP, Peloso 1901.06854

$$P_{\mathrm{model}}(k,\mu;b,A) = e^{-Ak^2}\left[\left(b + \mu^2 f R_{sd}(k)\right)^2 \left(P^{nw,l}(k) + P^{w,l}(k) e^{-k^2 \Xi^{rs}(\mu)/\gamma_{rec}}\right) + b^2 \Delta P^{nw,1l}_{\delta\delta}(k) + 2b\mu^2 f \Delta P^{nw,1l}_{\delta\theta}(k) + \mu^4 f^2 \Delta P^{nw,1l}_{\theta\theta}\right],$$

computed

	BOSS collaboration					
	Pre-reconstruction			Post-reconstruction		
	α	error	χ^2/gdl	α	error	χ^2/gdl
$0.2 < z < 0.5$	1.006	0.016	48.5/48	1.000	0.010	43.9/48
$0.4 < z < 0.6$	1.016	0.017	64.8/48	0.9936	0.0082	32.8/48
$0.5 < z < 0.75$	0.991	0.019	49.8/48	0.9887	0.0087	47.0/48
	Extractor procedure					
	Pre-reconstruction			Post-reconstruction		
	α	error	χ^2/gdl	α	error	χ^2/gdl
$0.2 < z < 0.5$	1.005	0.007	48/47	1.003	0.006	40/47
$0.4 < z < 0.6$	1.008	0.007	53/47	1.000	0.006	40/47
$0.5 < z < 0.75$	0.999	0.007	41/44	0.999	0.006	38/47

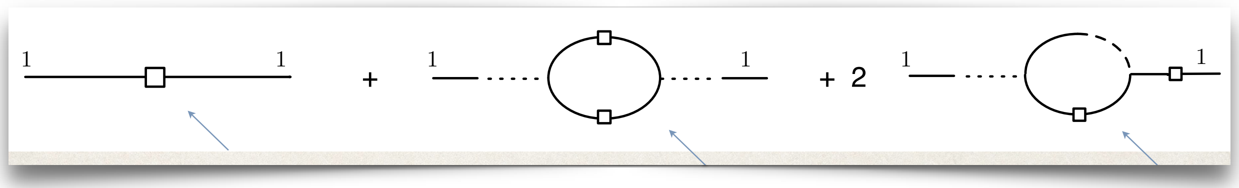
BOSS analysis (Beutler et al '16)

Noda, MP, Peloso 1901.06854

BOSS data already sensitive to BAO's at subpercent level!

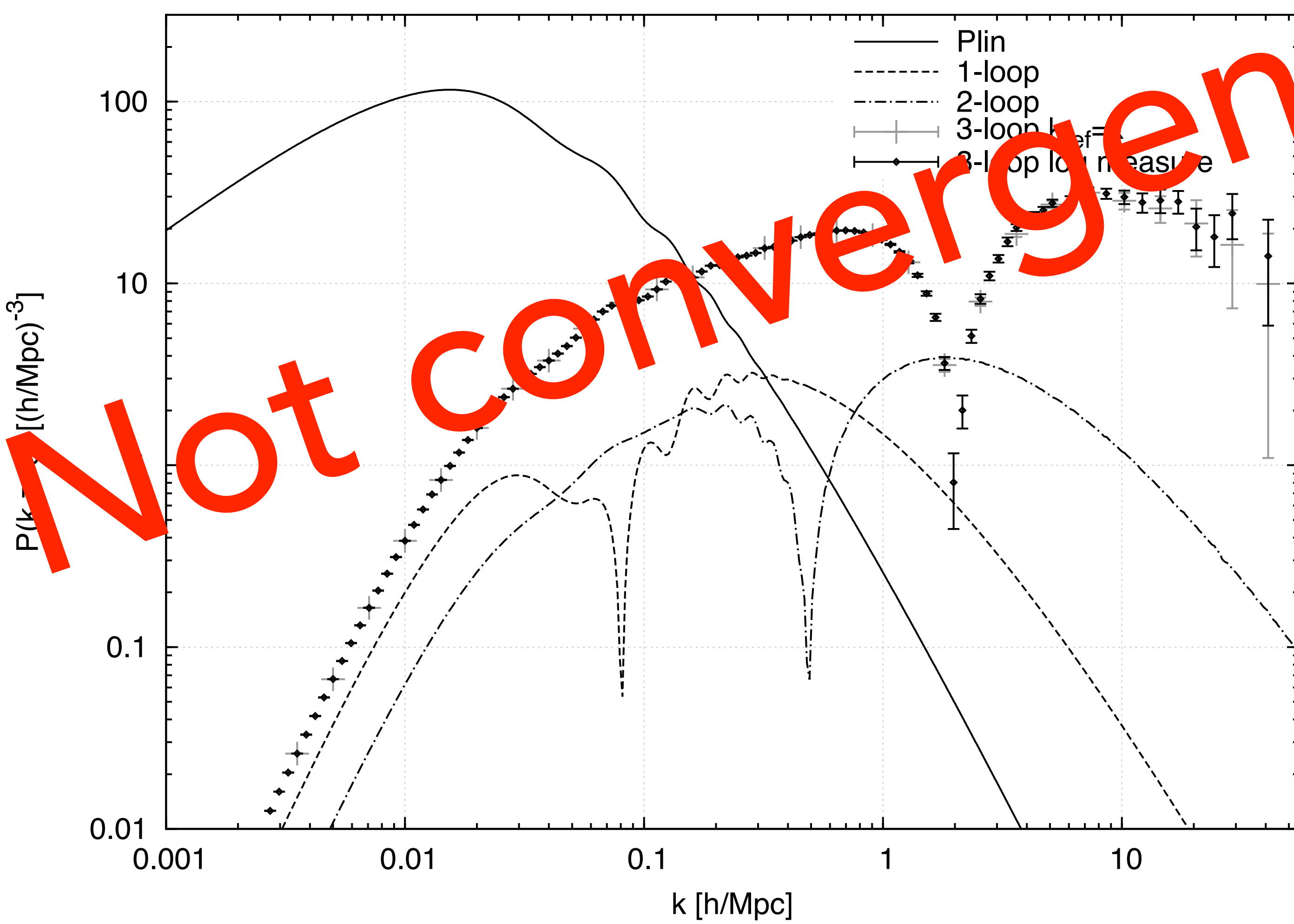
Performance of Standard PT

$$P(k,z) = D(z)^2 P^{(1)}(k) + D(z)^4 F^{(1l)}(k) + D(z)^6 F^{(2l)}(k) + \cdots$$

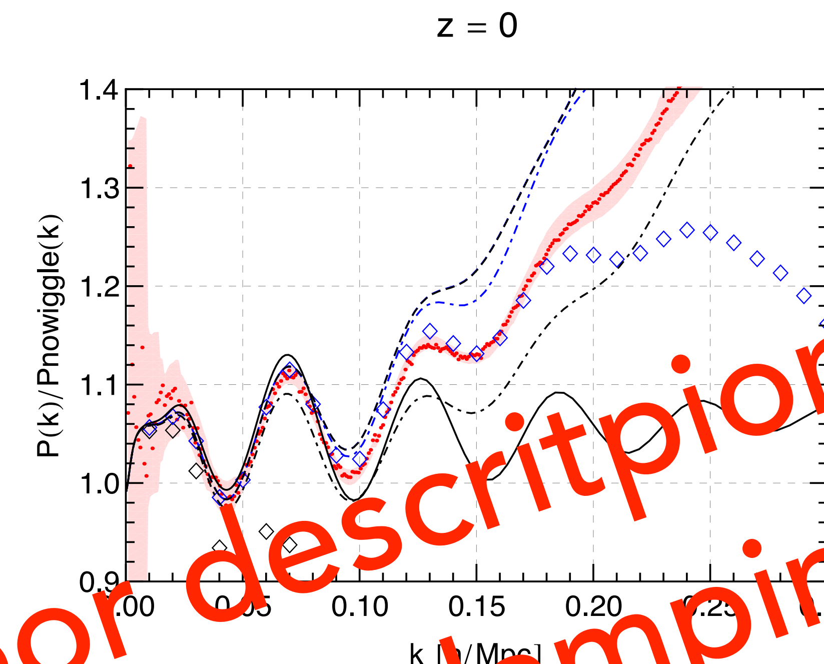


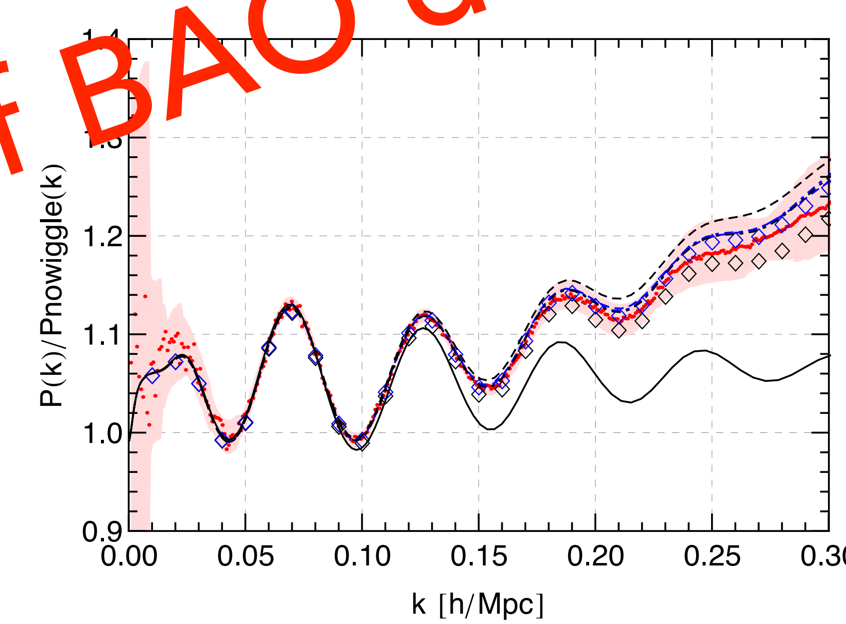
linear

1-loop



Not convergent !





poor descritpion of BAO damping

Blas et al. 1309.3308

MODE COUPLING

Linear Response Function

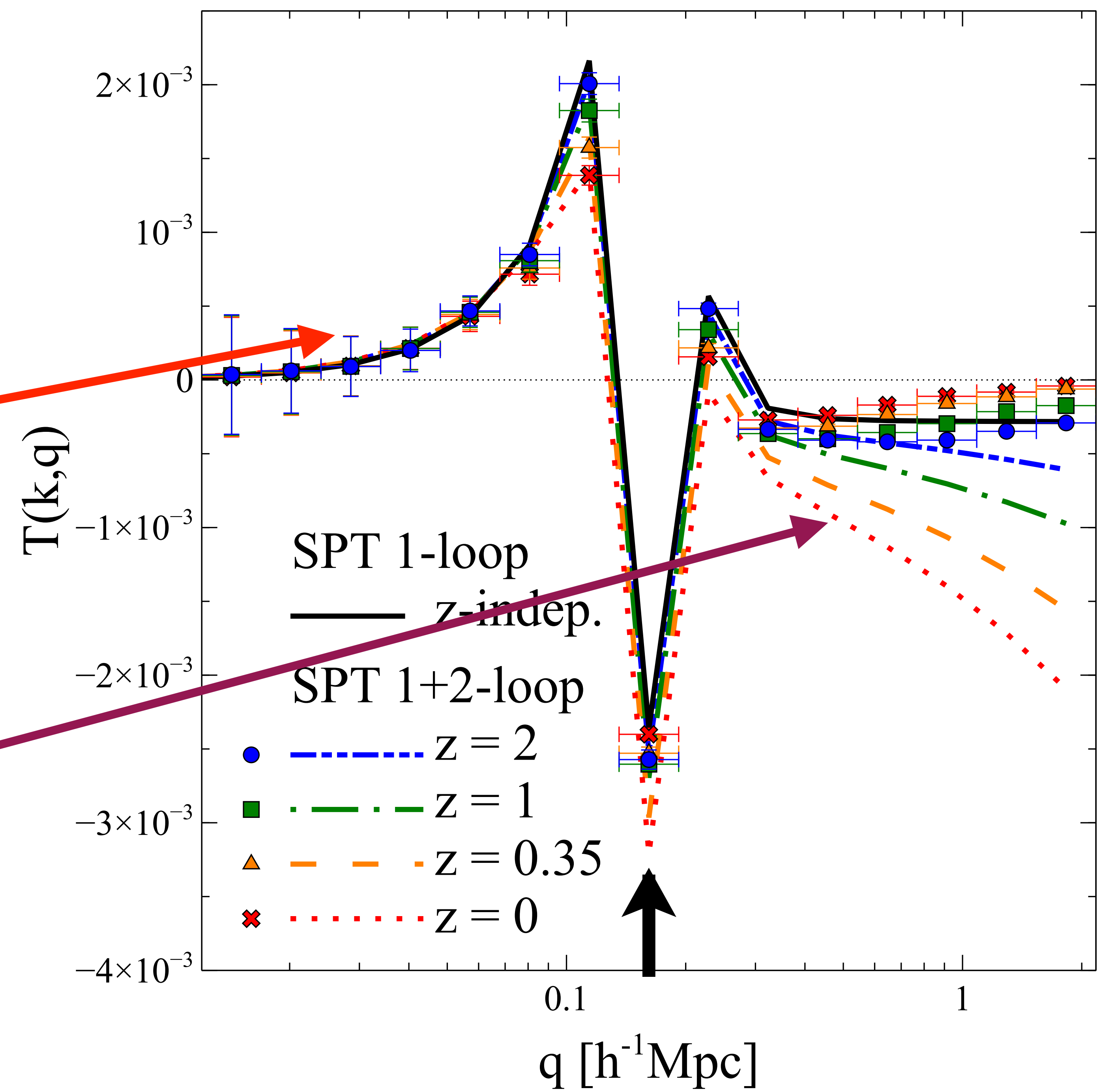
$$K(k,q;z) = q\frac{\delta P^{\rm nl}(k;z)}{\delta P^{\rm lin}(q;z)}$$

Nishimichi, Bernardeau, Taruya 1411.2970

IR: "Galilean" invariance (EP)

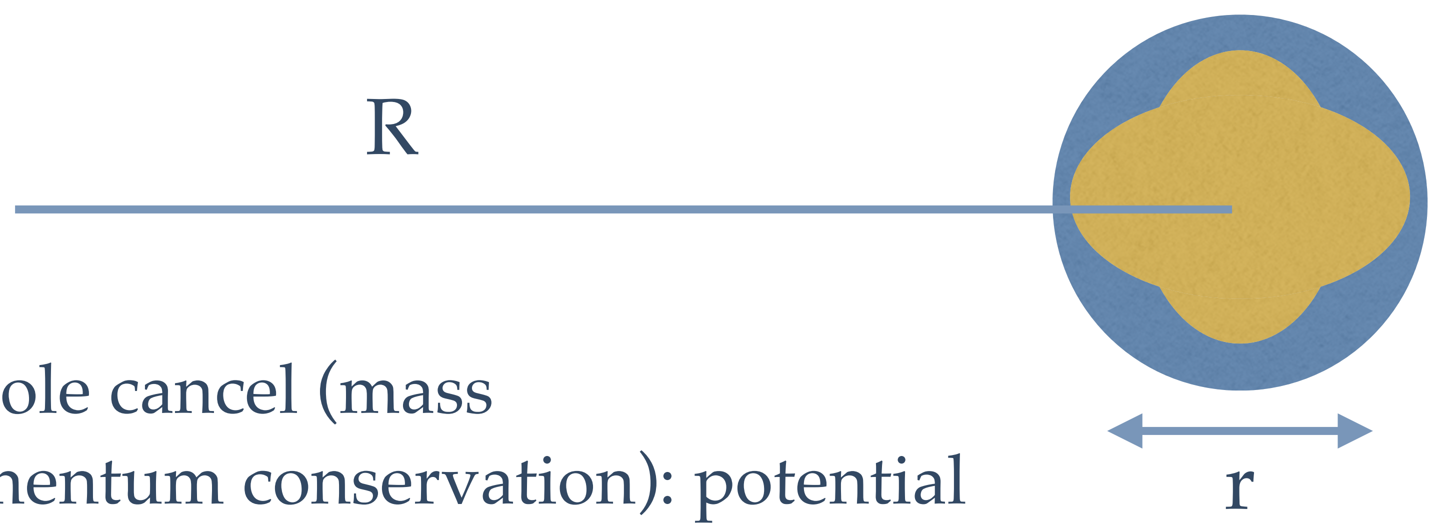
$$K(k,q;z) \sim q^3$$

UV: SPT over predicts the effect
of small scales on intermediate ones



The effect of short scales (UV)

Effect of an isolated small density profile ~ r at large distance R(>>r)



Monopole and dipole cancel (mass conservation+momentum conservation): potential generated by quadrupole:

$$\phi(R) \propto r^2/R^3 \qquad \delta \propto \nabla^2\phi(R) \to \delta(k) \propto k^2 \qquad P(k) \propto k^4$$

PT exhibits the k^4 decoupling of small scales q (k << q)

However, <u>virialized structures</u> decouple more efficiently than k^4 (Peebles '80, Baumann et al 1004.2488, Blas et al 1408.2995).
Highly nonlinear scales decouple.
<u>This effect is missed by the single stream approximation.</u>

Effective approaches to the UV

- Perturbation Theory (even after resummations) fails at short scales due to non-convergent series and multistreaming
- General idea: take the UV physics from N-body simulations and use PT only for the large and intermediate scales