

Diagrammatic techniques for soft photon bremsstrahlung

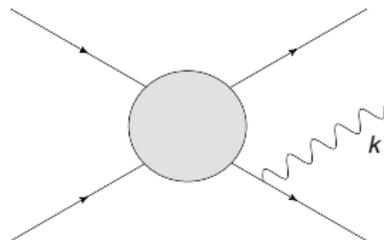
Domenico Bonocore

Institut für Theoretische Physik
Universität Münster

Soft photon Zoom-meeting
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INTRODUCTION

The study of the soft limit of gauge bosons radiation has a long history, dating back to the early days of quantum electrodynamics.



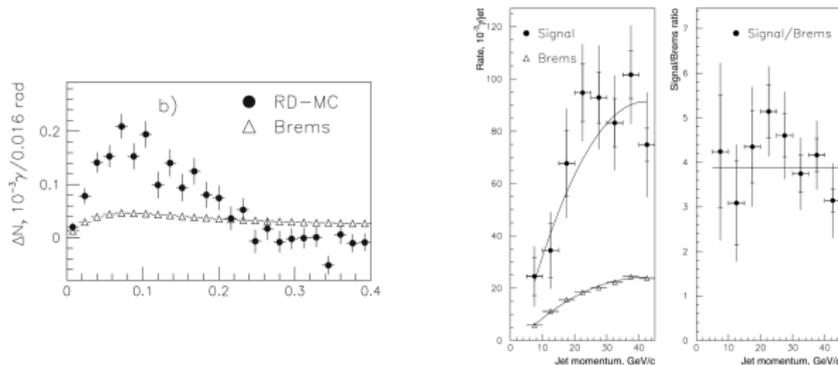
Recently a renewed interest in extending the strict soft limit $k \rightarrow 0$ (Leading Power (LP) in the soft expansion) to Next-to-Leading Power (NLP).

In particular, these effects might be sizable to measure the impact of undetected soft gluons that accompany hadronic initial and final states in scattering processes (**soft gluon resummation**). [Abbas, DB, Damsté, Laenen, Magnea, Vernazza, White 2015-2020, Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang 2016-2020, Moulton, Stewart, Tackmann, Vita, Zhu, 2015-2020,...]

INTRODUCTION

A similar set up is given by **soft photon bremsstrahlung**, where unlike the QCD case the soft bosons are detected.

Discrepancy in LEP data (DELPHI) w.r.t. the predicted LP bremsstrahlung spectrum when hadrons in final states [Abdallah et al. 2006-2010]



The existing description could be improved from many perspectives.

Goal of the talk is to show that the tools developed for perturbative QCD phenomenology at NLP can be naturally applied to the bremsstrahlung problem and might give important contributions.

OUTLINE

Soft Photons: Tree level (LP)

Soft Photons: Tree level (NLP)

Soft Photons: N soft emissions

Intermezzo: soft and collinear divergences to all orders

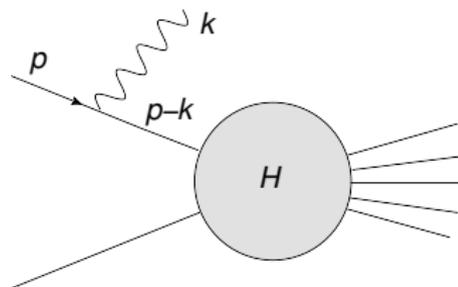
Soft photons: Loop Level

Conclusions

Soft Photons: Tree level (LP)

SOFT PHOTONS: TREE LEVEL (LP)

Consider the **Feynman diagram** for a photon emitted from an initial state fermion of charge Q . The interactions of the fermion with the other hard particles can be collected into the sub-diagram \mathcal{H} .



$$\mathcal{A}^\mu = \mathcal{H}(p-k) \frac{\not{p}-\not{k}}{(p-k)^2} (Q \gamma^\mu) u(p)$$

Then take the leading term for $k \rightarrow 0$ (**eikonal approximation**):

$$\mathcal{H}(p-k) = \mathcal{H}(p) + \mathcal{O}(k^0), \quad (p-k)^2 = -2p \cdot k + \mathcal{O}(k^0), \quad (1)$$

$$\not{p}\gamma^\mu u(p) = (2p^\mu - \gamma^\mu \not{p}) u(p) = 2p^\mu u(p) \quad (\text{Dirac eq.}) \quad (2)$$

Then the radiative amplitude \mathcal{A}^μ becomes proportional to the non-radiative amplitude \mathcal{A} :

$$\mathcal{A}^\mu = \left(-Q \frac{p^\mu}{p \cdot k} \right) \underbrace{\bar{u}(p) \mathcal{H}(p)}_{\mathcal{A}} + \mathcal{O}(k^0) = \text{wavy line} \times \text{sub-diagram H}$$

SOFT PHOTONS: TREE LEVEL (LP)

Summing over the n charged external legs of the amplitude \mathcal{A}_n (inserting a factor $\eta = \pm 1$ for incoming-outgoing particles), we get the **leading soft theorem**:

$$\epsilon_{\mu}^*(k)\mathcal{A}^{\mu} = \mathcal{S}_{LP}\mathcal{A}_n, \quad \mathcal{S}_{LP} = \sum_{i=1}^n Q_i\eta_i \frac{\epsilon^*(k) \cdot p_i^{\mu}}{p_i \cdot k} \quad (3)$$

- ▶ The photon interacts only via the eikonal rule $\frac{p^{\mu}}{p \cdot k}$: we lost information on the spin of the external legs (i.e. the charged emitters)
- ▶ hard particles do not recoil, because photon momentum $k \rightarrow 0$
- ▶ soft factor has decoupled from the hard dynamics, thus is insensitive to the short distance physics, i.e. soft photons cannot resolve details of the non radiative amplitude \mathcal{A}_n

SOFT PHOTONS: TREE LEVEL (LP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

The bremsstrahlung cross-section is obtained by squaring $\mathcal{S}_{LP}\mathcal{A}_n$ and integrating it over the phase space. Omitting constants with no relevance for this discussion (flux factor, coupling constants, powers of 2π , s , etc...):

$$\begin{aligned}
 \sigma_{LP} &\sim \int dPS \left| \begin{array}{c} \text{Diagram 1: } e^-(p) \text{ and } e^+(\bar{p}) \text{ meet at a vertex, } e^-(p-k) \text{ continues, and } \gamma(k) \text{ is emitted.} \\ \text{Diagram 2: } e^-(p) \text{ and } e^+(\bar{p}) \text{ meet at a vertex, } e^-(p) \text{ continues, and } \gamma(k) \text{ is emitted.} \end{array} \right|^2 \\
 &\sim \int dPS \left(\begin{array}{c} \text{Diagram 3: } \gamma(k) \text{ emitted from } e^-(p) \\ \text{Diagram 4: } \gamma(k) \text{ emitted from } e^+(\bar{p}) \end{array} \right)^2 \left| \begin{array}{c} \text{Diagram 5: } e^-(p) \text{ and } e^+(\bar{p}) \text{ meet at a vertex, } V(q) \text{ is produced.} \end{array} \right|^2 \\
 &\sim \int dPS \underbrace{\left(\frac{p^\mu}{p \cdot k} - \frac{\bar{p}^\mu}{\bar{p} \cdot k} \right)^2}_{\text{eikonal current}} |\mathcal{A}_2(p, \bar{p})|^2
 \end{aligned}$$

The non-radiative amplitude squared $|\mathcal{A}_2(p, \bar{p})|^2$ (i.e. the hard dynamics) is just a constant prefactor $\sim \text{Tr}[p\gamma^\alpha \bar{p}\gamma_\alpha]$ which does not depend on the final momenta. We omit it in the following.

SOFT PHOTONS: TREE LEVEL (LP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

The two-body phase space can be also expanded at LP:

$$\begin{aligned}\int d\text{PS} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3q}{(2\pi)^3 2\omega_q} (2\pi)^4 \delta^{(4)}(p + \bar{p} - k - q) \\ &= \int \frac{d^3k}{(2\pi)^2 4\omega_k \omega_q} \delta(\sqrt{s} - \omega_k - \omega_q) \\ &\xrightarrow{\omega_k \rightarrow 0} \frac{1}{4(2\pi)^2 \sqrt{s}} \int \frac{d^3k}{\omega_k} \delta(\sqrt{s} - \omega_q) \equiv \int d\text{PS}_{LP}\end{aligned}\quad (4)$$

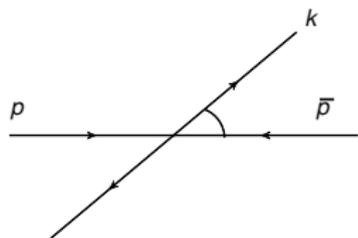
Thus (omitting constants)

$$\sigma_{LP} \sim \int \frac{d^3k}{\omega_k} \left(\frac{p^\mu}{p \cdot k} - \frac{\bar{p}^\mu}{\bar{p} \cdot k} \right)^2 \quad (5)$$

SOFT PHOTONS: TREE LEVEL (LP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

Calculation is easier in CM frame:



$$\begin{cases} p \cdot k = \omega_k(1 + \beta \cos(\theta)) \\ \bar{p} \cdot k = \omega_k(1 - \beta \cos(\theta)) \\ \beta = 1 - \frac{4m^2}{s} \\ \int dPS_{LP} \sim \int d\omega_k \omega_k \int d\theta \sin \theta \end{cases}$$

$$\sigma_{LP} \sim \int_{\omega_0}^{\omega_1} \frac{d\omega_k}{\omega_k} \int_{\theta_0}^{\theta_1} d\theta f(\theta) \quad \rightarrow \quad \frac{d^2\sigma}{d\omega_k d\theta} \sim \frac{1}{\omega_k} f(\theta) \quad (6)$$

$$f(\theta) = \sin \theta \left(\frac{1 - \frac{2m^2}{s}}{1 - \beta^2 \cos^2 \theta} - \frac{\frac{m^2}{s}}{(1 + \beta \cos \theta)^2} - \frac{\frac{m^2}{s}}{(1 - \beta \cos \theta)^2} \right) \quad (7)$$

- ▶ Number of photons $\frac{dN_\gamma}{d\omega_k} \sim \frac{d\sigma}{d\omega_k} \sim \frac{1}{\omega_k}$ (**soft singularity** for $\omega_0 \rightarrow 0$ in the total integrated cross-section σ_{LP})
- ▶ Power spectrum $I(\omega_k) = \frac{d\sigma_{LP}}{d\omega_k} \hbar\omega_k \sim \text{const.}$ It matches the analogous calculation in the classical theory

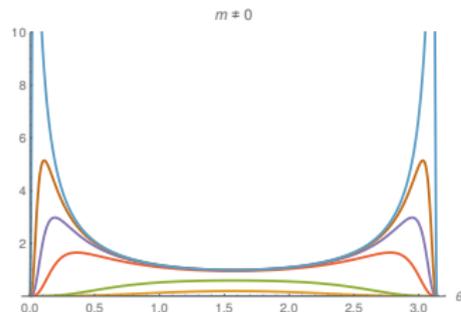
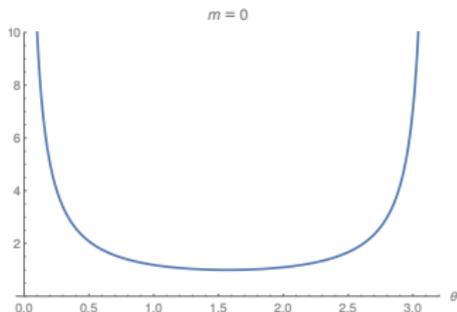
SOFT PHOTONS: TREE LEVEL (LP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

The **high energy limit** $s \gg m^2$ is achieved for $m \rightarrow 0$ (i.e. $\beta \rightarrow 1$)

$$\sigma_{LP} \stackrel{m=0}{\sim} \int_{\omega_0}^{\omega_1} d\omega_k \omega_k \int_{\theta_0}^{\theta_1} d\theta \sin\theta \frac{p \cdot \bar{p}}{p \cdot k \bar{p} \cdot k} \sim \int_{\omega_0}^{\omega_1} \frac{d\omega_k}{\omega_k} \int_{\theta_0}^{\theta_1} d\theta \frac{\sin\theta}{1 - \cos^2\theta}$$

- ▶ **Collinear singularity** for $\theta_0 \rightarrow 0$ in the total cross-section σ_{LP}
- ▶ **Angular distribution** $f(\theta)$ concentrated along beam axis at high energies (divergent for $m = 0$)
- ▶ The $m = 0$ is relevant for perturbative calculations in the parton model (massless quarks and gluons).



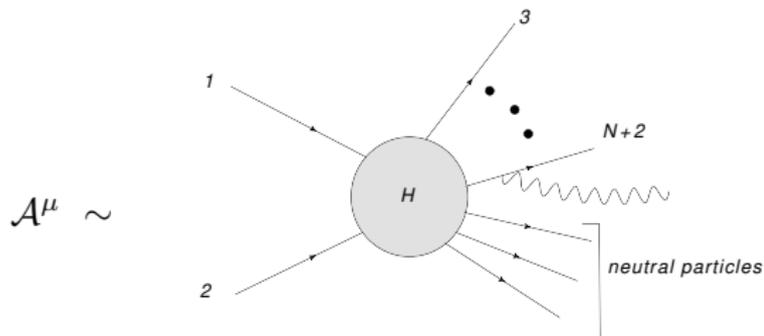
SOFT PHOTONS: TREE LEVEL (LP)

Generalizations:

- ▶ Analogous result for the production of additional N charged particles: we need to square $\mathcal{S}_{LP}\mathcal{A}_{N+2}$ and integrate it over the phase space, i.e.

$$\sigma_{LP} \sim \int \frac{d^3k}{\omega_k} \int \frac{d^3p_1}{\omega_{p_1}} \dots \int \frac{d^3p_N}{\omega_{p_N}} \sum_{i,j=1}^{N+2} \eta_i \eta_j \frac{p_i \cdot p_j}{(p \cdot k)(p_j \cdot k)} |\mathcal{A}_{N+2}(p_1, \dots, p_{N+2})|^2$$

- ▶ The emission of additional neutral elementary particles is irrelevant for soft bremsstrahlung (we factorize it and include it in \mathcal{H} , hence \mathcal{A})



Soft Photons: Tree level (NLP)

SOFT PHOTONS: TREE LEVEL (NLP)

At this point we would like to go beyond the strict soft (eikonal) approximation and expand at subleading power in the soft momentum.

Note: terminology for expansion of the soft theorem including subleading powers may vary in the literature:

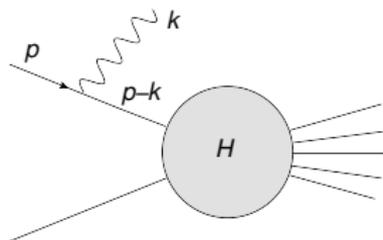
- ▶ next-to-next-to-...-soft $NN...S$
- ▶ sub-sub-...-leading
- ▶ next-to-next-to-...-eikonal $NN...E$
- ▶ next-to-next-to-...- leading power $NN...LP$

In this talk we consider only NLP.

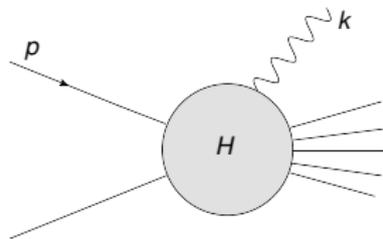
SOFT PHOTONS: TREE LEVEL (NLP)

From NLP the photon starts to be sensitive the hard dynamics, hence we allow the possibility that it is emitted from a virtual particle inside the subdiagram \mathcal{H} .

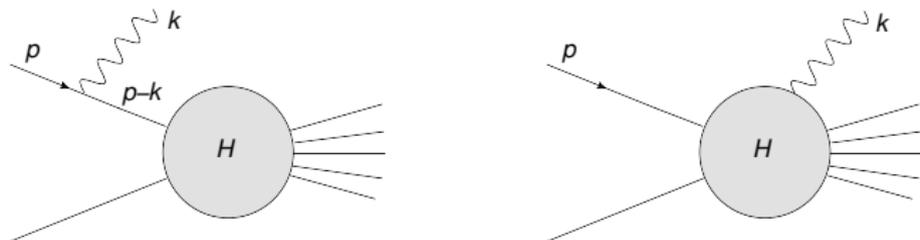
Therefore, we classify the diagrams in **external emissions** (already present at LP)



and **internal emissions**:



SOFT PHOTONS: TREE LEVEL (NLP)



- ▶ External emission: expand up to $\mathcal{O}(k)$

$$\begin{aligned}
 \mathcal{A}_{\text{ext}}^\mu(p) &= \mathcal{H}(p-k) \frac{(\not{p}-\not{k})}{(p-k)^2} (Q \gamma^\mu) u(p) & (8) \\
 &= Q \mathcal{H}(p) \left(\frac{p^\mu}{p \cdot k} + \frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{ik_\nu \sigma^{\mu\nu}}{p \cdot k} \right) u(p) \\
 &+ Q \frac{p^\mu}{p \cdot k} k^\nu \underbrace{\frac{\partial \mathcal{H}(p-k)}{\partial k_\nu} \Big|_{k=0}}_{-\frac{\partial \mathcal{H}(p)}{\partial p_\nu}} u(p) + \mathcal{O}(k)
 \end{aligned}$$

Here we used $\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$ where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is the Lorentz generator for particles of spin $\frac{1}{2}$. Then sum over all external legs

$$\mathcal{A}_{\text{ext}}^\mu = \sum_i \mathcal{A}_{\text{ext}}^\mu(p_i)$$

SOFT PHOTONS: TREE LEVEL (NLP)

- ▶ Internal emission: use Ward identity $k_\mu (\mathcal{A}_{\text{ext}}^\mu + \mathcal{A}_{\text{int}}^\mu) = 0$

$$\mathcal{A}_{\text{int}}^\mu = \sum_i Q_i \frac{\partial \mathcal{H}(p^i)}{\partial p_\mu^i} u(p^i) \quad (9)$$

- ▶ Adding $\mathcal{A}_{\text{ext}}^\mu$ and $\mathcal{A}_{\text{int}}^\mu$:

$$\begin{aligned} \mathcal{A}^\mu &= \sum_i Q_i \frac{p_i^\mu}{p_i \cdot k} \mathcal{A}(p_1 \dots p_n) \\ &+ \sum_i Q_i \left(\frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{ik_\nu \sigma^{\mu\nu}}{p \cdot k} \right) \mathcal{A}(p_1 \dots p_n) \\ &+ \sum_i Q_i \underbrace{\left(-\frac{p_i^\mu k^\nu}{p_i \cdot k} \frac{\partial}{\partial p_i^\nu} + \frac{\partial}{\partial p_i^\mu} \right)}_{\equiv L^{\mu\nu}} \mathcal{A}(p_1 \dots p_n) \end{aligned} \quad (10)$$

$L^{\mu\nu}$ is the angular momentum generator of the Lorentz group

SOFT PHOTONS: TREE LEVEL (NLP)

This is the **sub-leading** soft theorem, known as **Low's theorem**:

[Low 1958 (scalar emitters), Burnett-Kroll 1968 (spin $\frac{1}{2}$ emitters, conjecture for generic spin), Bell-VanRoyen 1969 (generic spin)]

For a real photon ($k^2 = 0$, $\epsilon(k) \cdot k = 0$) it has the more compact form

$$\epsilon_\mu^*(k) \mathcal{A}^\mu = (\mathcal{S}_{LP} + \mathcal{S}_{NLP}) \mathcal{A}_n, \quad (11)$$

$$\mathcal{S}_{LP} = \sum_{i=1}^n Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k}, \quad \mathcal{S}_{NLP} = \sum_{i=1}^n Q_i \frac{\epsilon_\mu^*(k) k_\nu (\sigma^{\mu\nu} + L^{\mu\nu})}{p_i \cdot k} \quad (12)$$

- ▶ corrections to the strict limit $k \rightarrow 0$: small recoil of the emitter taken into account
- ▶ sensitive to the spin of the emitter (e.g. $\sigma^{\mu\nu} = 0$ for scalars, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ for spin 1/2, etc.)
- ▶ orbital angular momentum $L^{\mu\nu}$ is sensitive to the short distance interactions in \mathcal{A} (hard lines do not start from a pointlike vertex)

SOFT PHOTONS: TREE LEVEL (NLP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

At **cross-section** level we have two sources of NLP corrections:

- ▶ NLP **matrix elements** (Low's theorem)
- ▶ NLP **phase space** corrections

$$\begin{aligned}\sigma_{NLP} = & \int dPS_{LP} \mathcal{M}_{LP} \mathcal{M}_{LP}^* + \int dPS_{LP} 2\text{Re}(\mathcal{M}_{NLP} \mathcal{M}_{LP}^*) \\ & + \int dPS_{NLP} \mathcal{M}_{LP} \mathcal{M}_{LP}^*\end{aligned}\quad (13)$$

where

$$\mathcal{M}_{LP} = \mathcal{S}_{LP} \mathcal{A} \quad (14)$$

$$\mathcal{M}_{NLP} = \mathcal{S}_{NLP} \mathcal{A} \quad (15)$$

SOFT PHOTONS: TREE LEVEL (NLP)

CASE STUDY: $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

The interference between \mathcal{S}_{NLP} and \mathcal{S}_{LP} , integrated over the LP phase space, yields

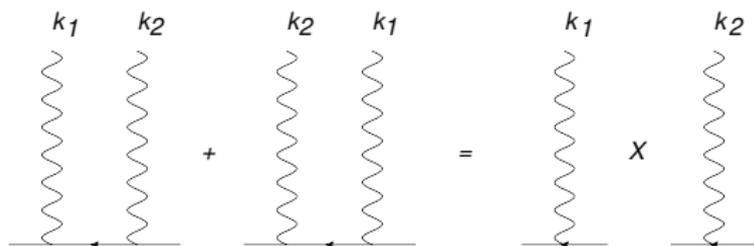
$$\begin{aligned}\sigma_{NLP} &\sim \sigma_{LP} + \int \frac{d^3k}{\omega} \left(\frac{p^\mu}{p \cdot k} \frac{\text{Tr}[\not{p}\gamma^\mu \not{k}\gamma^\alpha \not{\bar{p}}\gamma_\alpha]}{2\bar{p} \cdot k} + \frac{\bar{p}^\mu}{\bar{p} \cdot k} \frac{\text{Tr}[\not{p}\gamma^\alpha \not{\bar{p}}\not{k}\gamma^\mu \not{k}\gamma_\alpha]}{2p \cdot k} \right) \\ &\stackrel{m=0, \text{CM}}{\sim} \sigma_{LP} + \int_{\omega_0}^{\omega_1} d\omega \omega \int_{\theta_0}^{\theta_1} d\theta \sin \theta \left(\frac{1}{p \cdot k} + \frac{1}{\bar{p} \cdot k} \right) \\ &\sim \sigma_{LP} + \int_{\omega_0}^{\omega_1} d\omega \int_{\theta_0}^{\theta_1} \frac{d\theta \sin \theta}{1 - \cos^2 \theta}\end{aligned}\tag{16}$$

- ▶ same angular distribution as LP only because $m = 0$ (coll div).
- ▶ no $1/\omega$ factor, hence no soft divergence
- ▶ Number of photons $\frac{d\sigma}{d\omega} \sim \text{const}$, power spectrum
 $I(\omega) = \frac{d\sigma}{d\omega} \hbar\omega \sim \omega$

Soft Photons: N soft emissions

N SOFT EMISSIONS (LP)

Sum over the permutation of 2 emissions: eikonal identity



$$\frac{1}{(p \cdot k_1)(p \cdot (k_1 + k_2))} + \frac{1}{(p \cdot k_2)(p \cdot (k_1 + k_2))} = \frac{1}{p \cdot k_1} \frac{1}{p \cdot k_2} \quad (17)$$

Emissions factorize. Generalization to N -emissions is then trivial.

Factorization implies exponentiation: eikonal amplitudes can be generated by Wilson lines

$$\exp \left(i \int \frac{d^4 k}{(2\pi)^4} \frac{p^\mu}{p \cdot k} \tilde{A}_\mu(k) \right) = \exp \left(i \int_0^\infty d\lambda p^\mu A_\mu(\lambda x) \right) \quad (18)$$

Hard particle on the straight classical trajectory, it does not recoil.
Soft emissions affect it only by a phase.

N SOFT EMISSIONS (NLP)

Same strategy at NLP: sum over permutations but expand only one photon momentum at NLP (black dot). One gets a so-called next-to-eikonal identity:

The diagram illustrates the next-to-eikonal identity for two soft gluons. It is presented as an equation where the left side is a sum of four diagrams and the right side is a sum of four diagrams.

Left side (sum of four diagrams):

- Diagram 1: Two vertical wavy lines representing gluons with momenta k_1 and k_2 above them. A horizontal line with an arrow pointing left has a black dot at the left end. The k_1 gluon is attached to the dot, and the k_2 gluon is attached to the line to the right of the dot.
- Diagram 2: Similar to Diagram 1, but the k_2 gluon is attached to the dot and the k_1 gluon is attached to the line to the right.
- Diagram 3: Similar to Diagram 1, but the k_1 gluon is attached to the line to the right of the dot and the k_2 gluon is attached to the dot.
- Diagram 4: Similar to Diagram 1, but the k_2 gluon is attached to the line to the right of the dot and the k_1 gluon is attached to the dot.

Right side (sum of four diagrams):

- Diagram 5: Two separate vertical wavy lines with momenta k_1 and k_2 above them, each attached to its own horizontal line with an arrow pointing left.
- Diagram 6: Similar to Diagram 5, but the k_1 gluon is attached to a black dot on the horizontal line, and the k_2 gluon is attached to the line to the right of the dot.
- Diagram 7: Similar to Diagram 5, but the k_2 gluon is attached to a black dot on the horizontal line, and the k_1 gluon is attached to the line to the right of the dot.
- Diagram 8: A single vertex (black dot) where two wavy lines with momenta k_1 and k_2 meet. The vertex is attached to a horizontal line with an arrow pointing left.

Plus signs (+) are placed between the diagrams in both the left and right sums. The entire equation is preceded by an equals sign (=).

There are correlations between pairs of gluons. Naive factorization is broken (seagull vertex).

N SOFT EMISSIONS (NLP)

However, we find again exponentiation (also for off-shell contributions) [Laenen, Stavenga, White 2008, Laenen, Magnea, Stavenga, White 2010, DB 2020]

$$\begin{aligned} & \exp \left[\int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(k) \left(-\frac{p^\mu}{p \cdot k} + \frac{k^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} - \frac{i k_\nu \sigma^{\nu\mu}}{p \cdot k} \right) \right. \\ & + \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \tilde{A}_\mu(k) \tilde{A}_\nu(l) \left(\frac{\eta^{\mu\nu}}{2p \cdot (k+l)} - \frac{p^\nu l^\mu p \cdot k + p^\mu k^\nu p \cdot l}{2(p \cdot l)(p \cdot k)[p \cdot (k+l)]} \right. \\ & \left. \left. + \frac{(k \cdot l)p^\mu p^\nu}{2(p \cdot l)(p \cdot k)[p \cdot (k+l)]} - \frac{i\sigma^{\mu\nu}}{p \cdot (k+l)} \right) \right], \end{aligned} \quad (19)$$

Subleading power effects corresponds to the quantum fluctuation of the hard particle around the classical straight trajectory. Unlike Wilson line, angular momentum in the exponent.

Note that off-shell effects ($k^2 \neq 0, \epsilon \cdot k \neq 0$) are included.

TREE-LEVEL RECAP

- ▶ Amplitude level: tree level soft photon theorem

$$\epsilon_\mu^*(k) \mathcal{A}^\mu = (\mathcal{S}_{LP} + \mathcal{S}_{NLP}) \mathcal{A}_n, \quad (20)$$

$$\mathcal{S}_{LP} = \sum_{i=1}^n Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k}, \quad \mathcal{S}_{NLP} = \sum_{i=1}^n Q_i \frac{\epsilon_\mu^*(k) k_\nu J^{\mu\nu}}{p_i \cdot k}, \quad (21)$$

- ▶ Cross-section level: LO Bremsstrahlung $e^+(\bar{p})e^-(p) \rightarrow V(q)\gamma(k)$

$$\frac{d^2\sigma}{d\omega d\theta} = \frac{1}{\omega} \frac{d\sigma_{LP}}{d\theta} + \frac{d\sigma_{NLP}}{d\theta} + \mathcal{O}(\omega) \quad (22)$$

- ▶ Generalization to N soft photons possible both at LP and NLP

How do soft theorems (and the soft bremsstrahlung) change at loop-level?

Intermezzo: soft and collinear
divergences to all orders

WHERE DO IR DIVERGENCES COME FROM?

We already encountered soft and collinear divergences for phase space integrals: no problem for soft bremsstrahlung at tree level, due to finite resolution on energy and angle of the detected photon.



What about loop integrations? In that case we must integrate over *all* momentum configurations of the virtual particles: divergences are unavoidable in $D = 4$.

In particular, we expect soft theorems to relate infinite quantities

$$\underbrace{\mathcal{A}^\mu}_{\infty} = \underbrace{\mathcal{S}\mathcal{A}_n}_{\infty} \quad (23)$$

So, how can we compute physical observables (i.e. finite quantities)?
And what is the role of the soft theorem at loop level?
Answer: amplitudes are not observables!

WHERE DO IR DIVERGENCES COME FROM?

Solution at cross-section level: sum over the undetected photons/gluons. **Soft** divergences between loop and phase space integrations cancel (Bloch-Nordsieck mechanism).

Bremsstrahlung: kinematic cuts on detected photon (finite energy and angular resolution). Sum over undetected photons/gluons.

- ▶ At NLO in QED:

$$(e^+e^- \rightarrow q \bar{q} \gamma)_{1\text{loop}\gamma} + (e^+e^- \rightarrow q \bar{q} \gamma \gamma)_{\text{tree}}$$

- ▶ At NLO in QCD:

$$(e^+e^- \rightarrow q \bar{q} \gamma)_{1\text{loop}g} + (e^+e^- \rightarrow q \bar{q} \gamma g)_{\text{tree}}$$

With **hadrons**, perturbative calculations are carried by factorizing the non-perturbative dynamics and treating the initial/final states as **massless partons**. But with massless particles **collinear** sensitivity persists (Kinoshita-Lee-Nauenberg theorem). Then collinear divergences are absorbed into universal non-perturbative distribution functions (PDFs and Fragmentation Functions).

Hence, suitably defined cross-sections are finite.

WHERE DO IR DIVERGENCES COME FROM?

Still, the presence of IR divergences at the amplitude level modifies the NLP soft theorem non-trivially, and might have sizable effect at the cross-section level.

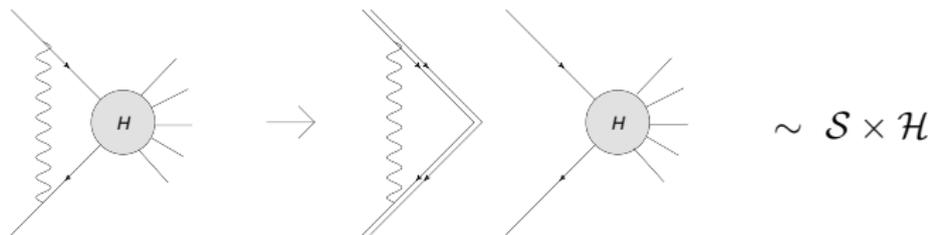
To understand why, we must first disentangle soft and collinear divergences in the loop diagrams without the extra radiated photon. [Landau 1959, Coleman-Norton 1965, Libby-Sterman 1978, Akhoury 1979, Sen 1983, Collins 1989, Beneke-Smirnov 1997]

Schematically, IR divergences factorize as

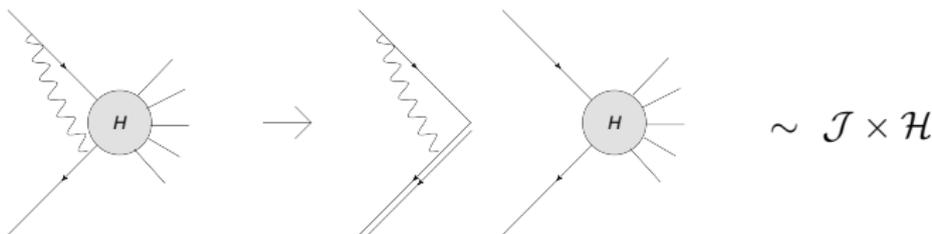
$$\underbrace{A}_{\infty} \sim \underbrace{\mathcal{J}(coll)}_{\infty} \times \underbrace{S(soft)}_{\infty} \times \mathcal{H}(hard) \quad (24)$$

SOFT-COLLINEAR FACTORIZATION

Soft:



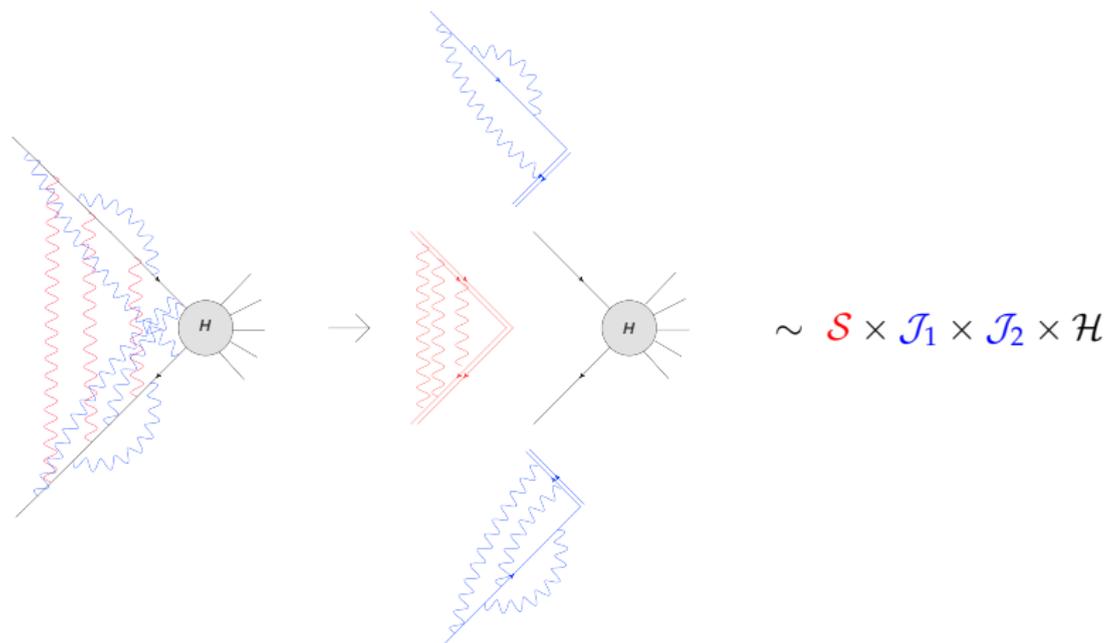
Collinear:



Here the double lines mean that the interaction is eikonal, i.e. with the combined vertex-propagator $\frac{p^\mu}{p \cdot k}$.

SOFT-COLLINEAR FACTORIZATION

The procedure can be generalized to all orders and leads to:



More information (and technicalities): [e.g. Sterman QFT book, Dixon, Magnea, Sterman 2008, Becher, Broggio, Ferroglia 2015]

Soft photons: Loop Level

SOFT PHOTONS - LOOP LEVEL (LP)

The soft theorems at loop level are obtained by attaching a real soft photon to the factorized diagrams. The strategy is

$$A^\mu = \mathcal{J}^\mu \times \mathcal{S} \times \mathcal{H} + \mathcal{J} \times \mathcal{S}^\mu \times \mathcal{H} + \mathcal{J} \times \mathcal{S} \times \mathcal{H}^\mu$$

Similarly to the tree level case, every time a photon is emitted from a virtual particle is a NLP effect. Thus at LP we should consider only the emission from the real external charged particles.

Diagrammatically, this means that the emission to consider is only from \mathcal{J} when it is **formed only by the real external lines**, with no collinear virtual particles. And this is precisely the external emission we considered at tree level.



SOFT PHOTONS - LOOP LEVEL (LP)

Hence

$$H^\mu(p) = \mathcal{O}(k^0) \quad (25)$$

$$S^\mu(p) = \mathcal{O}(k^0) \quad (26)$$

$$\mathcal{J}^\mu(p) = \mathcal{J}^\mu(p) \Big|_{\text{tree}} + \mathcal{O}(k^0) = Q \frac{p^\mu}{p \cdot k} + \mathcal{O}(k^0) \quad (27)$$

All virtual particles are then contained in \mathcal{J}_i , \mathcal{H} and \mathcal{S} and we get

$$\mathcal{A}^\mu = Q \frac{p^\mu}{p \cdot k} \underbrace{\mathcal{J}_1 \dots \mathcal{J}_n \mathcal{S} \mathcal{H}}_{\mathcal{A}_n} \quad (28)$$

Thus the **LP soft theorem holds also at the loop level**

$$\epsilon^*(k)_\mu \mathcal{A}^\mu = \mathcal{S}_{LP} \mathcal{A}_n, \quad \mathcal{S}_{LP} = \sum_i Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k} \quad (29)$$

and the corresponding bremsstrahlung distribution is the same as the tree level one.

SOFT PHOTONS - LOOP LEVEL (NLP) - MASSIVE

At NLP we have to include the radiative functions \mathcal{H}^μ , \mathcal{S}^μ and \mathcal{J}^μ .

The discussion is different for massive and massless (or small mass) particles. Let us first assume that the external charged particles are **massive**, and that the mass is not small, i.e. the soft momentum k is the only small quantity in the process.

Then there are **no collinear particles**, and the Jet functions \mathcal{J}_i are just composed by the external real charged particles like at LP. The radiative function \mathcal{J}^μ reduces once again to the external emission:

$$\mathcal{J}^\mu(p) = \mathcal{J}^\mu(p) \Big|_{\text{tree}} + \mathcal{O}(k) = \quad (30)$$

$$= Q \left(\frac{p^\mu}{p \cdot k} + \frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{ik_\nu \sigma^{\mu\nu}}{p \cdot k} \right) + \mathcal{O}(k) \quad (31)$$

SOFT PHOTONS - LOOP LEVEL (NLP) - MASSIVE

Then, the contribution from \mathcal{S}^μ or \mathcal{H}^μ can be derived from the external emission via Ward identities (as we did at tree level) and the result matches the tree level result (i.e. we get the orbital angular momentum $L^{\mu\nu}$ contribution).

$$\epsilon_\mu^*(k)\mathcal{A}^\mu = (\mathcal{S}_{LP} + \mathcal{S}_{NLP})\mathcal{A}_n, \quad (32)$$

$$\mathcal{S}_{LP} = \sum_i Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k}, \quad \mathcal{S}_{NLP} = \sum_i Q_i \frac{\epsilon_\mu^*(k) k_\nu J^{\mu\nu}}{p_i \cdot k} \quad (33)$$

Hence, for $\omega_k \ll \frac{m^2}{Q}$, **Low's theorem receives no corrections** at loop-level.

However, the region $\omega_k \ll \frac{m^2}{Q}$ is very narrow at high energies. And it does not include the massless region, which is needed for perturbative calculations of hadronic external states (which need massless partons)

SOFT PHOTONS - LOOP LEVEL (NLP) - MASSLESS

The larger region $\frac{m^2}{Q} \ll \omega_k \sim m$ (which includes the massless limit) was first considered by Del Duca [Nucl.Phys.B 345 (1990)].

For this kinematical region \mathcal{J} contains **loops of virtual collinear particles**. Thus

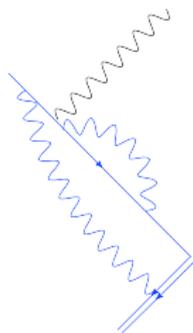
$$\mathcal{J}^\mu = \mathcal{J}^\mu|_{\text{tree}} + \mathcal{J}^\mu|_{\text{loop}} \quad (34)$$

The \mathcal{H}^μ and \mathcal{S}^μ can be computed once again via Ward identities in terms of the external emission $\mathcal{J}^\mu|_{\text{tree}}$.

SOFT PHOTONS - LOOP LEVEL (NLP) - MASSLESS

The collinear contribution in $\mathcal{J}^\mu|_{\text{loop}}$ instead requires a separate computation.

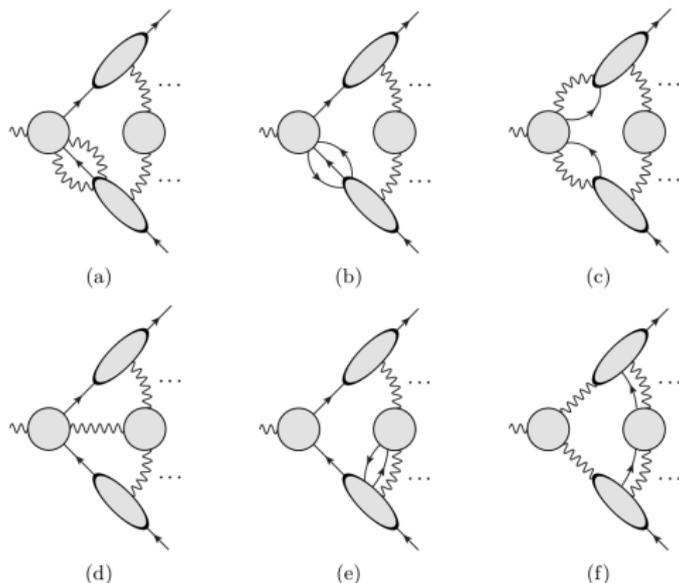
The reason is that in the **collinear region** we are not allowed to Taylor expand in k (as we did for $\mathcal{J}^\mu|_{\text{tree}}$, H^μ and S^μ) since k is not the only small quantity in the process (also $m \rightarrow 0$).



Thus we need to include this new object $\mathcal{J}^\mu|_{\text{loop}}$, called **radiative jet function**, which must be computed separately. This has been done at one loop [DB, Laenen, Magnea, Melville, Vernazza, JHEP 06 (2015) 008]

SOFT PHOTONS - LOOP LEVEL (NLP) - MASSLESS

In the previous slides we schematically depicted an emission from loops containing collinear virtual particles by \mathcal{J}^μ . A more careful power counting analysis (at all-loops) reveals that at order $\mathcal{O}(k^0)$ several types of (radiative) jets are needed: [Larkoski, Neill, Stewart, 2014, Gervais 2017, Moutl, Stewart, Vita, 2019, Laenen, Damsté, Vernazza, Waalewijn, Zoppi, 2020]



REVISITING THE DELPHI ANALYSIS

- ▶ Current theoretical description is based on **LP soft theorems**.
- ▶ **NPL** contributions from **Low's theorem** and **phase space**, although suppressed by a power of ω_k , are still tree-level contributions. Hence, their effect might be sizable.
[DB, Kulesza, Stüber, in progress]
- ▶ For muon production at LEP ($m \sim 100$ MeV, $Q \sim 100$ GeV), Low's theorem can be safely applied only to photons with very small energies $\omega_k \ll \frac{m^2}{Q} \sim 0.1$ MeV. Moreover, when hadrons are present, perturbative calculations in the parton model requires massless partons (quarks and gluons). Hence, Low's theorem must be supplemented with **radiative jets**. Although they are suppressed by powers of α_{em} or α_s , they provide corrections whose numerical impact must be investigated.
[DB, Kulesza, in progress]

Conclusions

CONCLUSIONS

- ▶ The study of IR divergences in the **soft-collinear factorization** approach is a mature field. Recent progress in extending the framework at NLP for QCD phenomenology.
- ▶ These tools can be naturally applied to the bremsstrahlung problem, where the spectrum is governed by **soft theorems**
- ▶ **LP** soft theorem is universal, insensitive to spin and recoil, and valid at all-loop order.
- ▶ **NLP** corrections to LP soft theorem:
 - ▶ **Phase space** corrections with LP matrix elements.
 - ▶ **Low's theorem**: exact at tree level (at loop level is exact only if $\omega_k \ll \frac{m^2}{Q}$, a very narrow region in the high energy limit!). Sensitive to angular momentum and recoil of the charged particles.
 - ▶ **Radiative jet functions**: needed at loop level when $\frac{m^2}{Q} \ll \omega_k \sim m$ (collinear effects of the virtual particles). All-order structure more intricate, but under control.
- ▶ **N** soft photon emissions at tree level under control to all-orders both at LP and NLP, off-shell contributions are included