FUNDAMENTAL PHYSICS WITH GRAVITATIONAL waveS

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Passo del Tonale

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Basic Concepts

- Need a mathematical formalism based on GR
- Derive rigorously (gauge choices) GW equations
  in multiple contexts (Mink. / FLRW / Schw.)
- Origin, propagation and detection opens
  a door into fundamental physics (GR, QM, TD)

Linearization of GR

GW interact weakly → linearize GR

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Diffeomorphism invariance

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

Small perturbation around empty space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \qquad |h_{\mu\nu}| \ll 1$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Linearization of GR

Linearize the Riemann tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\Big(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\Big)$$

Introduce $\bar{h}$

$$h = \eta^{\mu\nu} h_{\mu\nu} \qquad \bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

Substitute in the full Einstein Equations

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Linearization of GR

Residual freedom, choose Lorentz gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$

$$\bar{h}_{\mu\nu} \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\rho \xi^\rho$$

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$$ where $$\Box = \eta_{\mu\nu}\partial^\mu \partial^\nu = \partial_\mu \partial^\mu$$

$$\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x) \quad \Longrightarrow \quad \Box \xi_\mu = f_\mu(x)$$

Final result (with sources)

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

in vacuum

$$\Box \bar{h}_{\mu\nu} = 0$$

Linearization of GR

Use the gauge to remove spurious degrees of freedom

How many d.o.f. does a GW have?

$$\Box \xi_\mu = 0 \qquad h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Make the choices:

$$\xi^0(x) \implies \bar{h} = 0 = h$$

$$\xi^i(x) \implies h^{0i}(x) = 0$$

Eliminate some of the h_{ij} :

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \implies \partial^0 h_{00} + \partial^i h_{0i} = 0 \implies \partial^0 h_{00} = 0$$

Finally the TT gauge:

$$\boxed{h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad \partial^j h_{ij} = 0}$$

Plane wave solutions

Plane wave solutions in vacuum: $k^\mu = (\frac{\omega}{c}, \vec{k})$, with $k_\mu k^\mu = 0 \implies \omega = c|\vec{k}|$

$$\Box \bar{h}_{\mu\nu} = 0 \implies h_{ij}^{\mathrm{TT}}(x) = e_{ij}(\mathbf{k})\, e^{ik\cdot x}$$

$$\hat{n} = \vec{k}/|\vec{k}| \overset{\partial^j h_{ij}=0}{\Longrightarrow} n^i h_{ij} = 0$$

Polarizations:

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)]$$

Space-time element:

$$ds^2 = -c^2 dt^2 + dz^2 + (1 + h_+ \cos[\omega(t - z/c)])dx^2 + (1 - h_+ \cos[\omega(t - z/c)])dy^2 + 2h_\times \cos[\omega(t - z/c)]dxdy$$

Expansion in Fourier space:

$$h_{ij}(t,\mathbf{x}) = \sum_{\lambda=+,\times} \int_{-\infty}^{\infty} df \int d^2\hat{\mathbf{n}}\, \tilde{h}_\lambda(f,\hat{\mathbf{n}})\, e_{ij}^\lambda(\hat{\mathbf{n}})\, e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)}$$

$$e_{ij}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ij}$$

$$e_{ij}^\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij}$$

GW effect on matter

Geodetic deviation:

$$\left[x^\mu(\tau),\, x^\mu(\tau) + \xi^\mu(\tau)\right] \qquad \frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \quad \Longrightarrow \quad \ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\,\xi^j$$

The + polarization:

$$h_{ij}^{\mathrm{TT}} = h_+ \sin\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ij}$$

$$\delta\ddot{x} = -\frac{h_+}{2}(x_0 + \delta x)\,\omega^2 \sin\omega t \quad \Longrightarrow$$

$$\delta x(t) = +\frac{h_+}{2}\,x_0\,\sin\omega t$$

$$\delta\ddot{y} = +\frac{h_+}{2}(y_0 + \delta y)\,\omega^2 \sin\omega t \quad \Longrightarrow$$

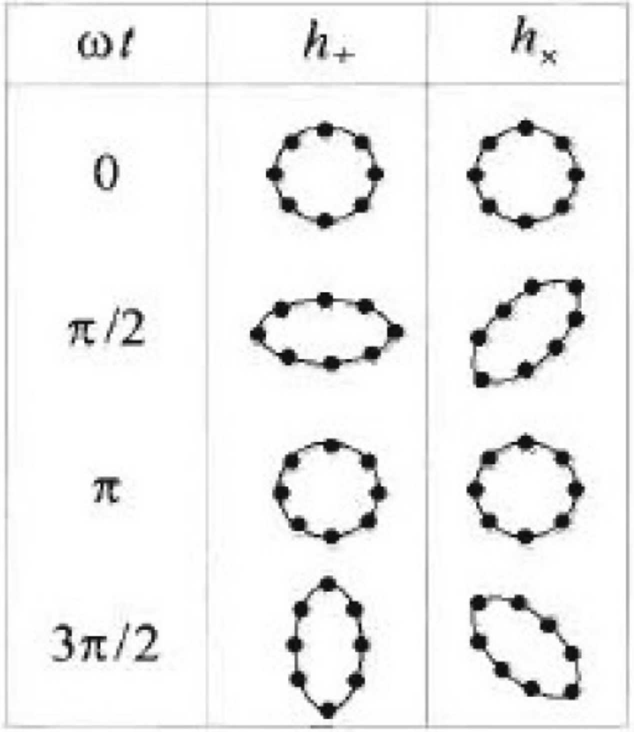
$$\delta y(t) = -\frac{h_+}{2}\,y_0\,\sin\omega t$$

The x polarization:

$$h_{ij}^{\mathrm{TT}} = h_\times \sin\omega t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij}$$

$$\delta x(t) = +\frac{h_\times}{2}\,y_0\,\sin\omega t$$

$$\delta y(t) = +\frac{h_\times}{2}\,x_0\,\sin\omega t$$



GW energy & momentum

Expansion to second order:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots$$

Einstein equations averaged over wavelengths/periods

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad \Longrightarrow \quad \bar{R}_{\mu\nu} = -\left\langle R^{(2)}_{\mu\nu}\right\rangle + \frac{8\pi G}{c^4}\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle$$

Define energy-momentum tensor of the GW

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{(2)}\right\rangle \quad \Longrightarrow \quad \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)$$

GW energy & momentum

Expansion:

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu}$$
$$+h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h_\nu{}^\alpha\partial_\beta h_{\mu\alpha} - \partial^\beta h_\nu{}^\alpha \partial_\alpha h_{\mu\beta}$$
$$-\partial_\beta h^{\alpha\beta}\partial_\nu h_{\mu\alpha} + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\nu\alpha}$$
$$+\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - \frac{1}{2}\partial^\alpha h\,\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h\,\partial_\nu h_{\alpha\mu} + \frac{1}{2}\partial^\alpha h\,\partial_\mu h_{\alpha\nu}\Big]$$

GW energy-momentum tensor:

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\,\partial_\nu h^{\alpha\beta}\right\rangle \quad\Longrightarrow\quad t^{00} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle \qquad E_V = \int_V d^3x\, t^{00}$$

Energy flux and momentum carried by the waves:

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G}\int d\Omega\left\langle \dot{h}^{\mathrm{TT}}_{ij}\dot{h}^{\mathrm{TT}}_{ij}\right\rangle \qquad \frac{dP^k}{dt} = -\frac{c^3 r^2}{32\pi G}\int d\Omega\left\langle \dot{h}^{\mathrm{TT}}_{ij}\partial^k h^{\mathrm{TT}}_{ij}\right\rangle$$

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle \qquad J^i = \frac{c^2}{32\pi G}\int d^3x\left[-\epsilon^{ijk}\left\langle \dot{h}^{\mathrm{TT}}_{ab}x^j\partial^k h^{\mathrm{TT}}_{ab}\right\rangle + 2\epsilon^{ijk}\left\langle h^{\mathrm{TT}}_{aj}\dot{h}^{\mathrm{TT}}_{ak}\right\rangle\right]$$

GW solutions in matter

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \Longrightarrow \quad \bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' \, G(x-x') T_{\mu\nu}(x')$$

$$\Box_x G(x-x') = \delta^4(x-x') \qquad G(x-x') = -\frac{\delta(x^0_{\text{ret}} - x'^0)}{4\pi|\mathbf{x}-\mathbf{x}'|} \qquad t_{\text{ret}} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$$

GW solution in matter: $\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \, \frac{T_{\mu\nu}(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$

Post-Newtonian expansion: $|\mathbf{x}-\mathbf{x}'| = r - \hat{\mathbf{n}} \cdot \mathbf{x}' + \mathcal{O}\left(\frac{d^2}{r}\right)$

$$T_{ij}\left(t - \frac{r}{c} + \frac{\hat{\mathbf{n}} \cdot \mathbf{x}'}{c}, \mathbf{x}'\right) \simeq T_{ij}\left(t - \frac{r}{c}\right) + \frac{n^k x'^k}{c} \partial_0 T_{ij} + \frac{1}{2c^2} x'^k x'^l n^k n^l \partial_0^2 T_{ij} + \dots$$

GW solutions in matter

Moments of the matter distribution

$$M = \frac{1}{c^2}\int d^3x\, T^{00}(t,\,\mathbf{x})$$
$$M^i = \frac{1}{c^2}\int d^3x\, T^{00}(t,\,\mathbf{x})\, x^i$$
$$M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\,\mathbf{x})\, x^i x^j$$
$$M^{ijk} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\,\mathbf{x})\, x^i x^j x^k$$

$$\rho = \frac{1}{c^2}T^{00}$$

$$M^{ij} = \left(M^{ij} - \frac{1}{3}\delta^{ij}M_{kk}\right) + \frac{1}{3}\delta^{ij}M_{kk}$$

$$[h_{ij}^{\mathrm{TT}}(t,\,\mathbf{x})]_{\mathrm{quad}} = \frac{2G}{c^4 r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\,\ddot{M}^{kl}(t - r/c)$$

Quadrupole tensor: $$Q^{ij} = M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} = \int d^3x\, \rho(t,\,\mathbf{x})\left(x^i x^j - \frac{1}{3}r^3\delta^{ij}\right)$$

GW solution in matter: $$\boxed{[h_{ij}^{\mathrm{TT}}(t,\,\mathbf{x})]_{\mathrm{quad}} = \frac{2G}{c^4 r}\,\ddot{Q}_{ij}^{\mathrm{TT}}(t - r/c)}$$

GW solutions in matter

Radiated energy - Quadrupole:

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{c^3 r^2}{32\pi G}\left\langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right\rangle$$

$$\boxed{P_{\text{quad}} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij} \right\rangle}$$

$$J^i = \frac{c^2}{32\pi G}\int d^3x \left[-\epsilon^{ijk}\left\langle \dot{h}_{ab}^{\text{TT}} x^j \partial^k h_{ab}^{\text{TT}} \right\rangle + 2\epsilon^{ijk}\left\langle h_{aj}^{\text{TT}} \dot{h}_{ak}^{\text{TT}} \right\rangle\right]$$

$$\left(\frac{dJ^i}{dt}\right)_{\text{quad}} = \frac{2G}{5^5}\epsilon^{ijk}\left\langle \ddot{Q}_{ja}\dddot{Q}_{ka} \right\rangle$$

Radiated energy - Octupole:

$$\mathcal{M}^{ijk} = M^{ijk} - \frac{1}{5}\left(\delta^{ij}M^{llk} + \delta^{ik}M^{ljl} + \delta^{jk}M^{ill}\right)$$

$$\left(h_{ij}^{\text{TT}}\right)_{\text{oct}} = \frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\, n_m\, \dddot{\mathcal{M}}^{klm}\,(t - r/c)$$

$$P = \frac{G}{c^5}\left[\frac{1}{5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij} \right\rangle + \frac{1}{189\, c^2}\left\langle \ddddot{\mathcal{M}}_{ijk}\ddddot{\mathcal{M}}_{ijk} \right\rangle\right]$$

Black Hole Binaries

Circular orbits: $\omega_s^2 = \dfrac{GM}{R^3}$

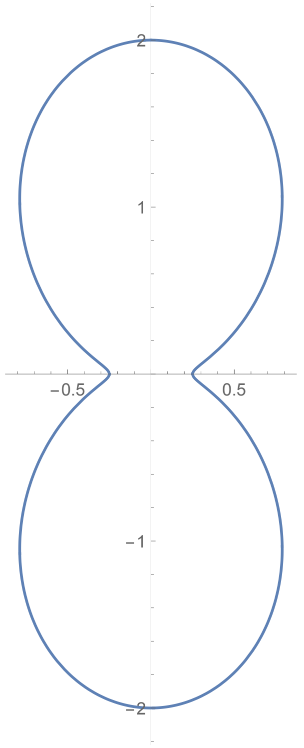
$$h_+(t;\theta,\phi) = \frac{4G\mu\,\omega_s^2 R^2}{c^4\,r}\,\frac{1+\cos^2\theta}{2}\,\cos(2\omega_s t_{\rm ret} + 2\phi)$$

$$\begin{aligned} x_0(t) &= R\,\cos(\omega_s t) \\ y_0(t) &= R\,\sin(\omega_s t) \\ z_0(t) &= 0 \end{aligned}$$

$$h_\times(t;\theta,\phi) = \frac{4G\mu\,\omega_s^2 R^2}{c^4\,r}\,\cos\theta\,\sin(2\omega_s t_{\rm ret} + 2\phi)$$

Power emitted:

$$\omega = 2\omega_s$$



$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{c^3 r^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle = \frac{2G\mu^2\,\omega_s^6 R^4}{\pi\,c^5}\,g(\theta)$$

$$g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta$$

$$P_{\rm quad} = \frac{32G\mu^2\,\omega_s^6 R^4}{5\,c^5} = \frac{G\mu^2\,\omega^6 R^4}{10\,c^5}$$

Black Hole Binaries

Chirp mass:

$$h_+(t;\theta,\phi) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

$$M_c = \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$$h_\times(t;\theta,\phi) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}}{c}\right)^{2/3}\cos\theta\,\sin(2\pi f_{\text{gw}} t_{\text{ret}} + 2\phi)$$

$$\omega_{\text{gw}} = 2\pi f_{\text{gw}}$$

Loss of energy in GW:

$$\dot{R} = -\frac{2}{3}R\frac{\dot{\omega}_s}{\omega_s}$$

$$P = \frac{32c^5}{5G}\left(\frac{GM_c}{c^2}\cdot\frac{\pi f_{\text{gw}}}{c}\right)^{10/3}$$

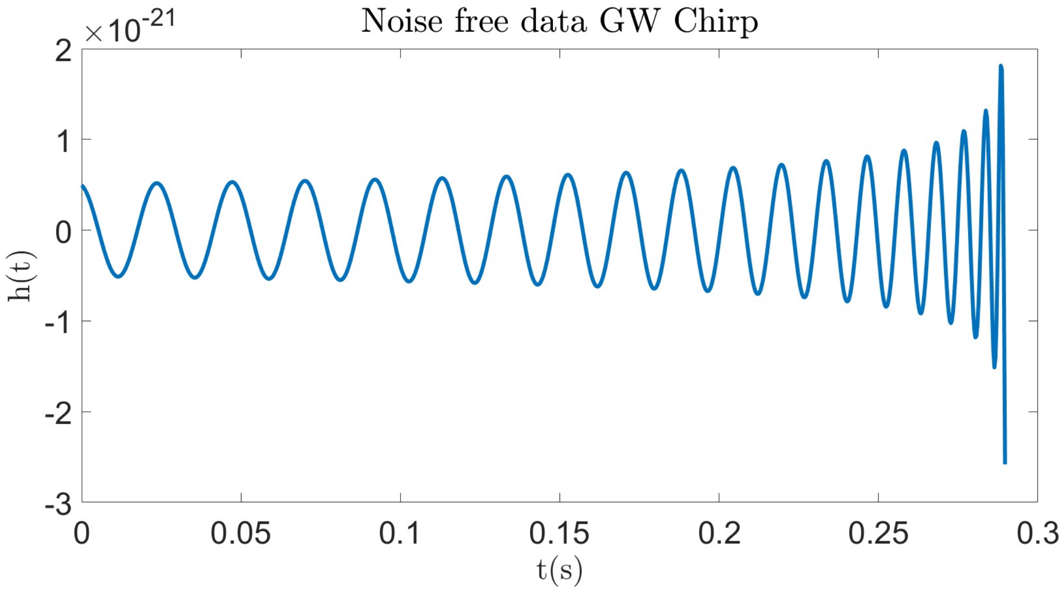
$$E = E_{\text{kin}} + E_{\text{pot}} = -\frac{Gm_1m_2}{2R} = -\left(\frac{G^2M_c^5\omega_{\text{gw}}^2}{32}\right)^{1/3} \implies \dot{\omega}_{\text{gw}} = \frac{12}{5}\,2^{1/3}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega_{\text{gw}}^{11/3}$$

Black Hole Binaries

Time of coalescence: $\tau = t_{\rm coal} - t$

$$\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3} \quad \Longrightarrow \quad f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256\,\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8} \simeq 134\,\mathrm{Hz}\left(\frac{1.21\,M_\odot}{M_c}\right)^{5/8}\left(\frac{1\,s}{\tau}\right)^{3/8}$$

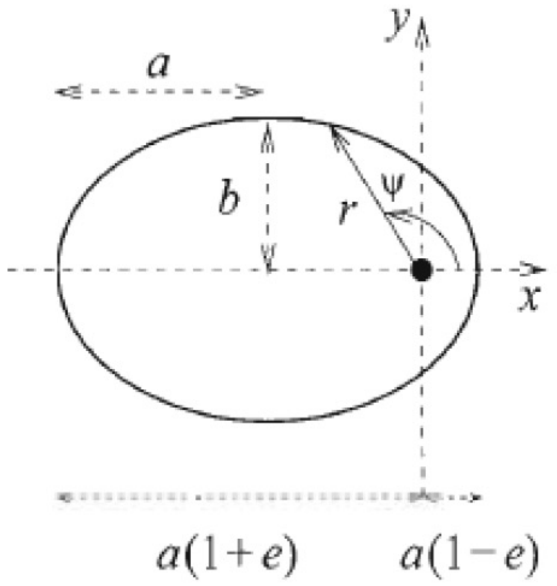
Chirp in amplitude:



Black Hole Binaries

Elliptical orbits:

$$e^2 = 1 + \frac{2EL^2}{G^2M^2\mu^3} < 1$$



$$\dot{\psi} = \frac{(GMR)^{1/2}}{r^2} \qquad M_{ij} = \mu\, r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}$$

$$r = \frac{a(1-e^2)}{1 + e\,\cos\psi}$$

Radiated power:

$$P(\psi) = \frac{G}{5\,c^5}\left[\dddot{M}_{11}^{\,2} + \dddot{M}_{22}^{\,2} + 2\,\dddot{M}_{12}^{\,2} - \frac{1}{3}(\dddot{M}_{11} + \dddot{M}_{22})^2\right]$$

$$= \frac{8G}{15\,c^5}\frac{G^3M^3\mu^2}{a^5(1-e^2)^5}(1 + e\,\cos\psi)^4\left[12(1 + e\,\cos\psi)^2 + e^2\sin^2\psi\right]$$

Black Hole Binaries

Average per orbit:

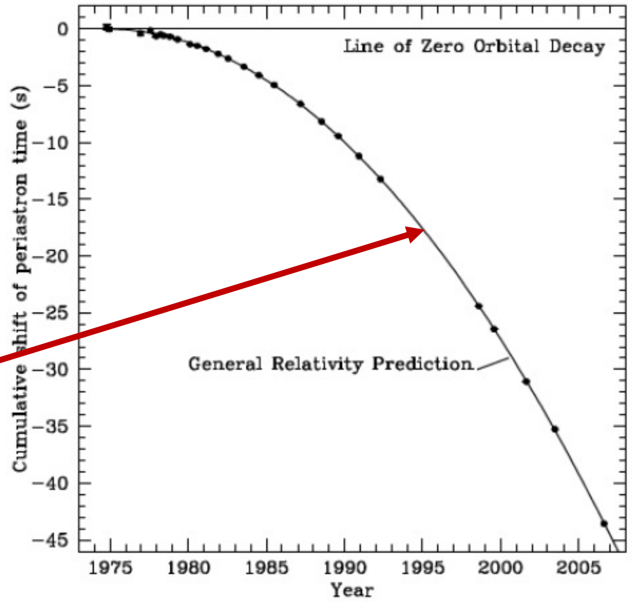
$$P = \frac{1}{T}\int_0^T dt\, P(\psi) = \frac{8G^4M^3\mu^2}{15\,c^5a^5}(1-e^2)^{-7/2}\int_0^{2\pi}\frac{d\psi}{2\pi}\left[12(1+e\,\cos\psi)^4 + e^2(1+e\,\cos\psi)^2\sin^2\psi\right]$$

$$P = \frac{32G^4M^3\mu^2}{5\,c^5a^5}f(e) \qquad f(e) = (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

Energy loss → change in period:

$$a = \frac{GM\mu}{2|E|} \qquad T \propto (-E)^{-3/2}$$

$$\frac{\dot{T}}{T} = -\frac{96c}{5}\left(\frac{GM}{c^2}\right)^{2/3}\frac{G\mu}{c^2}\left(\frac{cT}{2\pi}\right)^{-8/3} f(e)$$



Black Hole Binaries

Time evolution:

$$\frac{dE}{dt} = -\frac{32\,G^4M^3\mu^2}{5\,c^5a^5}\,f(e)$$

$$\frac{dL}{dt} = -\frac{32\,G^{7/2}M^{5/2}\mu^2}{5\,c^5a^7/2}\,\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

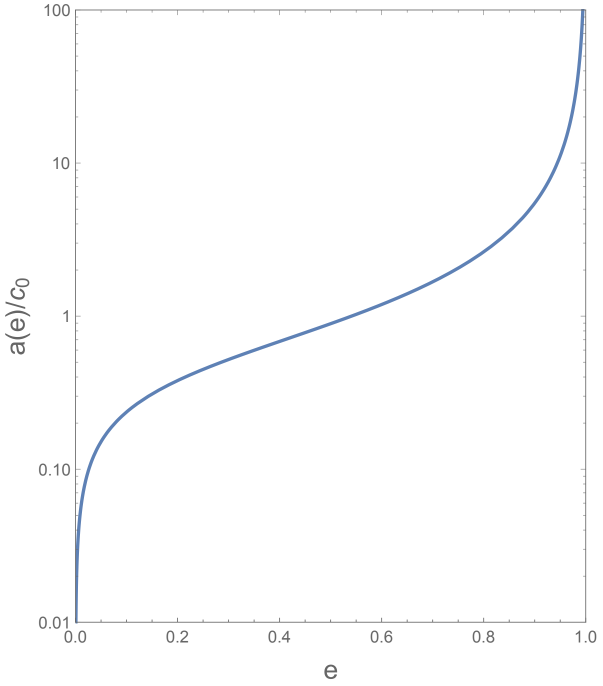
$$\Longrightarrow$$

$$\frac{da}{dt} = -\frac{64\,G^3M^2\mu}{5\,c^5a^3}\,f(e)$$

$$\frac{de}{dt} = -\frac{304\,G^3M^2\mu}{15\,c^5a^4}\,\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Circularization of the orbit:

$$\frac{da}{de} = \frac{12\,a}{19}\,\frac{1+(73/24)e^2+(37/96)e^4}{e(1-e^2)[1+(121/304)e^2]} \Longrightarrow a(e) = c_0\frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$



Time to coalescence:

$$\tau(a_0,e_0) = \frac{15\,c^5}{304\,G^3M^3\mu}\int_0^{e_0} de\,\frac{a^4(e)(1-e^2)^{5/2}}{e(1+(121/304)e^2)}$$

$$\simeq 9,829\,\mathrm{Myr}\left(\frac{T_0}{1\,\mathrm{hr}}\right)^{8/3}\left(\frac{M_\odot}{M}\right)^{2/3}\left(\frac{M_\odot}{\mu}\right)F(e_0)$$

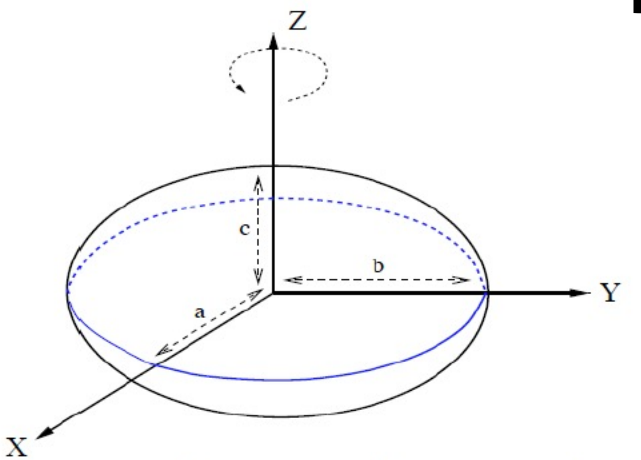
$$m_1 = m_2 \simeq 1.4\,M_\odot$$

$$T_0 \simeq 8\,\mathrm{hr}$$

$$\Downarrow$$

$$\tau(a_0,e_0) \simeq 300\,\mathrm{Myr}$$

Rotating Quasispherical NS



$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = 1$$

Tensor of Inertia:

$$I_{ij} = \int_V d^3x\, \rho\, (r^2 \delta_{ij} - x_i x_j) \quad \Longrightarrow \quad Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij} I\right)$$

$$I_{ij} = \frac{M}{5} \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Change to a rotating frame w.r.t. z-axis:

$$x_i = R_{ij}\, x'_j \qquad \varphi = \Omega\, t$$

$$R_{ij} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{ij} = R_{ik} R_{jl} I'_{kl} = (R I' R^{\mathrm{T}})_{ij}$$

$$= \begin{pmatrix} I_1 \cos^2\varphi + I_2 \sin^2\varphi & (I_1 - I_2)\sin\varphi\cos\varphi & 0 \\ (I_1 - I_2)\sin\varphi\cos\varphi & I_1 \sin^2\varphi + I_2 \cos^2\varphi & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Rotating Quasispherical NS

Quadrupole Tensor:

$$Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij} I\right) = -I_{ij} + \text{const.}$$

$$\operatorname{Tr} I = I_1 + I_2 + I_3 = \text{const.}$$

$$\Longrightarrow \quad Q_{ij} = \frac{I_2 - I_1}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + \text{const.}$$

For spherically symmetric NS:

$$I_2 = \frac{M}{5}(a^2 + c^2)$$

$$I_1 = \frac{M}{5}(b^2 + c^2)$$

$$\Longrightarrow \quad Q_{ij} = 0$$

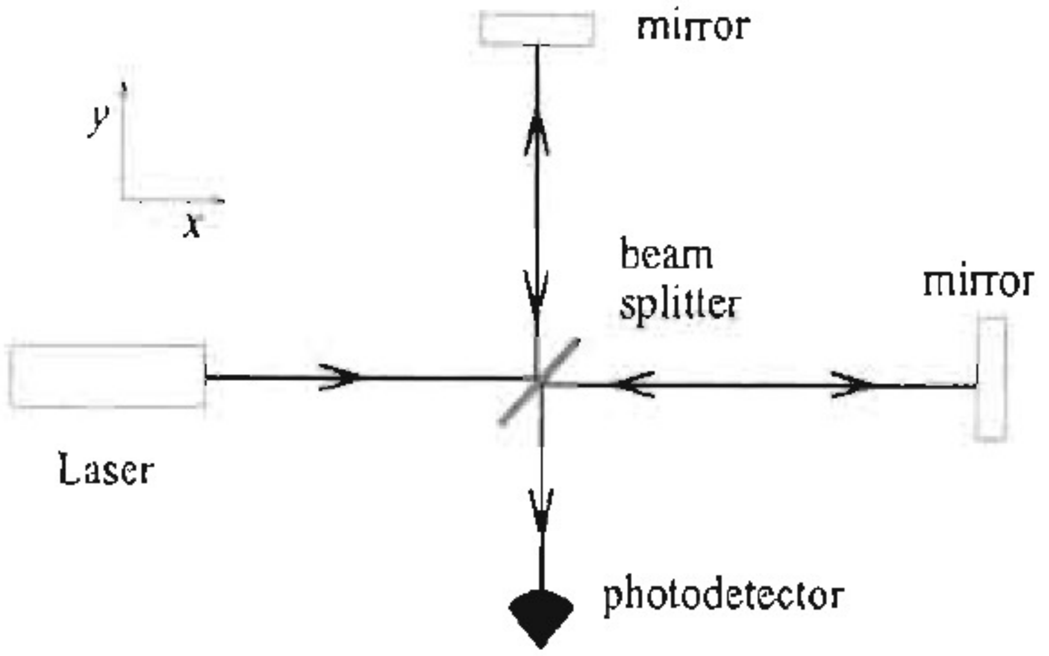
In general:

$$\epsilon \equiv \frac{2(a-b)}{(a+b)} \quad \Longrightarrow \quad \frac{I_2 - I_1}{I_3} = \frac{\epsilon}{2}\,\frac{(a+b)^2}{a^2 + b^2} = \epsilon + \mathcal{O}(\epsilon^3)$$

$$Q_{ij} = \frac{\epsilon\, I_3}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + \text{const.} \quad \Longrightarrow \quad L_{\mathrm{GW}} = \frac{32\,G}{5\,c^5}\Omega^6 I_3^2 \epsilon^2$$

Detection of GW

Laser interferometer:



Electric field:

$$E_1 = -\frac{1}{2}E_0\, e^{-i\omega_L t + 2ik_L L_x}$$

$$E_2 = +\frac{1}{2}E_0\, e^{-i\omega_L t + 2ik_L L_y}$$

$$E_{\rm out} = E_1 + E_2 = -iE_0\, e^{-i\omega_L t + ik_L(L_x+L_y)} \sin[k_L(L_y - L_x)]$$

$$|E_{\rm out}|^2 = E_0^2 \sin^2[k_L(L_y - L_x)]$$

Detection of GW

Effect of GW (+ pol) on distances: $h_+(t) = h_0 \cos \omega_{\rm gw} t$

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]dx^2 + [1 - h_+(t)]dy^2 + dz^2 \quad \overset{ds^2=0}{\Longrightarrow} \quad dx = \pm c dt \left[1 - \frac{1}{2} h_+(t)\right]$$

$$L_x = c(t_1 - t_0) - \frac{c}{2}\int_{t_0}^{t_1} dt'\, h_+(t')$$ + on the way back $$L_x = c(t_2 - t_1) - \frac{c}{2}\int_{t_1}^{t_2} dt'\, h_+(t')$$

Total time and difference in phase:

$$t_2 - t_0 = \frac{2L_x}{c}\left[1 + \frac{h_0(t_0 + L_x/c)}{2}\frac{\sin(\omega_{\rm gw} L_x/c)}{\omega_{\rm gw} L_x/c}\right]$$

$$\Delta\phi_x(t) = h_0 \frac{\omega_L L_x}{c}\frac{\sin(\omega_{\rm gw} L_x/c)}{\omega_{\rm gw} L_x/c}\cos[\omega_{\rm gw}(t - L_x/c)]$$

$$P = P_0 \sin^2[\phi_0 + \Delta\phi_x(t)]$$

$$\Delta\phi_{\rm Mich} = \Delta\phi_x - \Delta\phi_y = 2\Delta\phi_x$$

$$\Delta P_{GW} = \frac{P_0}{2}|\sin 2\phi_0|\Delta\phi_{\rm Mich}$$

Detection of GW

The detector measures the total strain: $h(t) = D^{ij} h_{ij}(t)$

The measurement depends on the transfer function: $\bar{h}_{\rm out}(f) = T(f)\bar{h}(f)$

The output also includes the detector noise:
(Assumed Gaussian and Stationary)

$$s_{\rm out}(t) = h_{\rm out}(t) + n_{\rm out}(t)$$

$$\langle n(t) \rangle = 0\,, \qquad \delta(f=0) \to \left[\int_{-T/2}^{T/2} dt\, e^{2\pi i f t}\right]_{f\to 0} = T$$

$$\langle \tilde{n}^*(f)\tilde{n}(f') \rangle = \frac{S_n(f)}{2}\,\delta(f-f') \Longrightarrow \langle |\tilde{n}(f)|^2 \rangle = \frac{1}{2} S_n(f)\,T$$

Detection of GW

Properties of the noise (Gaussian and Stationary):

$$\Delta f = 1/T$$

$$\frac{1}{2}S_n(f) = \langle|\tilde{n}(f)|^2\rangle \Delta f \qquad \Longrightarrow \qquad \langle n^2(t)\rangle = \langle n^2(t=0)\rangle = \frac{1}{2}\int_{-\infty}^{\infty} df\, S_n(f) = \int_0^{\infty} df\, S_n(f)$$

Signal to Noise Ratio. (K = filter function):

$$\langle n(t)\rangle = 0$$

$$S = \int_{-\infty}^{\infty} dt\, \langle s(t)\rangle\, K(t) = \int_{-\infty}^{\infty} dt\, h(t)\, K(t) = \int_{-\infty}^{\infty} df\, \tilde{h}(f)\tilde{K}^*(f)$$

$$N^2 = \langle s^2(t)\rangle_{h=0} = \int\int_{-\infty}^{\infty} dt\, dt'\, K(t)\, K(t')\langle n(t)n(t')\rangle = \int_{-\infty}^{\infty} df\, \frac{1}{2}S_n(f)|\,\tilde{K}(f)|^2$$

Detection of GW

Signal to Noise Ratio:

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f)\tilde{K}^*(f)}{[\int_{-\infty}^{\infty} df\, (1/2)S_n(f)|\tilde{K}(f)|^2]^{1/2}}$$

Optimal filter:

$$\tilde{K}(f) = \text{const.}\, \frac{\tilde{h}(f)}{S_n(f)} \Longrightarrow \boxed{\left(\frac{S}{N}\right)^2 = 4\int_0^{\infty} df\, \frac{|\tilde{h}(f)|^2}{S_n(f)}}$$

Example 1: Stochastic GW Background

$$\langle h_{ij}(t)h^{ij}(t)\rangle = 4\int_0^{\infty} df\, S_h(f)$$

$$\rho_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle \Longrightarrow \boxed{\Omega_{\rm gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f} = \frac{4\pi^2}{3H_0^2}f^3 S_h(f)}$$

$$\rho_{\rm gw} = \int_0^{\infty} d(\log f)\frac{d\rho_{\rm gw}}{d\log f}$$

Detection of GW

Example 2: Distance to coalescing binaries

$$\tilde{h}(f) = \frac{\sqrt{5/6}}{2\,\pi^{2/3}}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6} f^{-7/6}\, e^{i\Psi}\, Q(\theta, \phi; i)$$

Depends on the geometry of the system, inclination, etc.

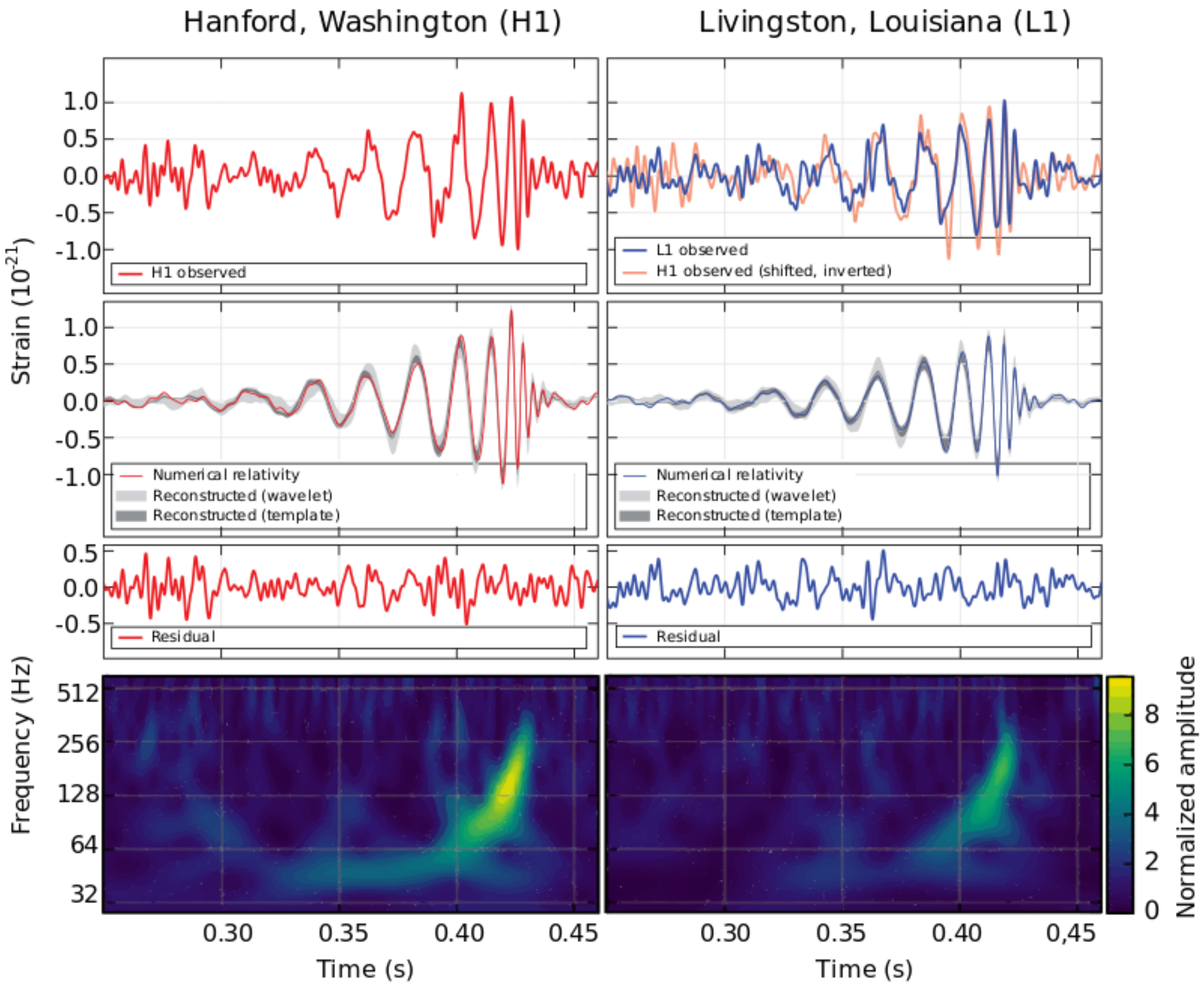
$$\Longrightarrow \quad \left(\frac{S}{N}\right)^2 = \frac{5}{6\pi^{4/3}}\frac{c^2}{r^2}\left(\frac{GM_c}{c^3}\right)^{5/3} |Q(\theta, \phi; i)|^2 \int_0^{f_{\mathrm{max}}} df\, \frac{f^{-7/3}}{S_n(f)}$$

Averaging over inclination, we can solve for the distance

$$d_{\mathrm{sight}} = \frac{2c}{5(S/N)}\frac{\sqrt{5/6}}{\pi^{2/3}}\left(\frac{GM_c}{c^3}\right)^{5/6}\left[\int_0^{f_{\mathrm{max}}} df\, \frac{f^{-7/3}}{S_n(f)}\right]^{1/2}$$

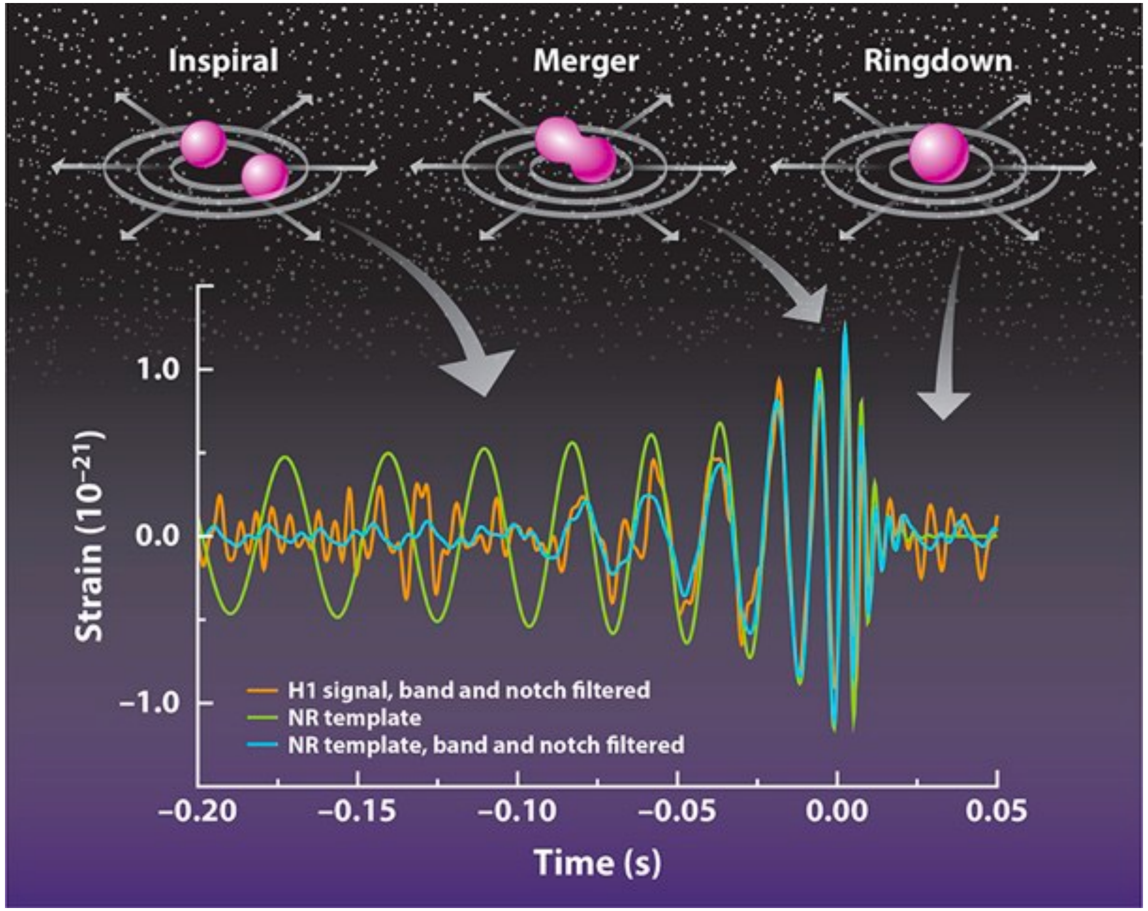
Observation of GW

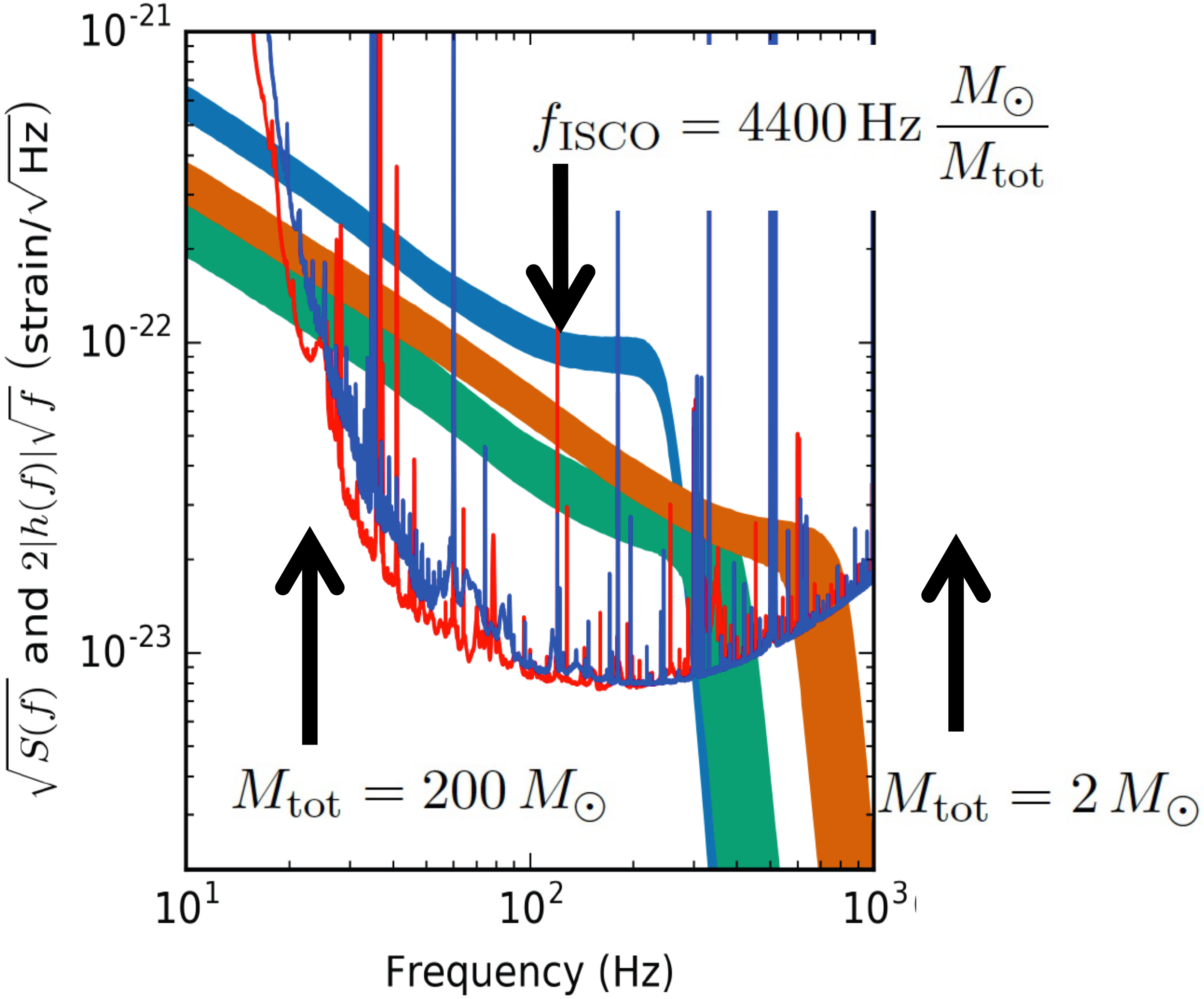
First **direct detection** of GW (14 Sep 2015): GW150914



Observation of GW

First **direct detection** of GW (14 Sep 2015): GW150914





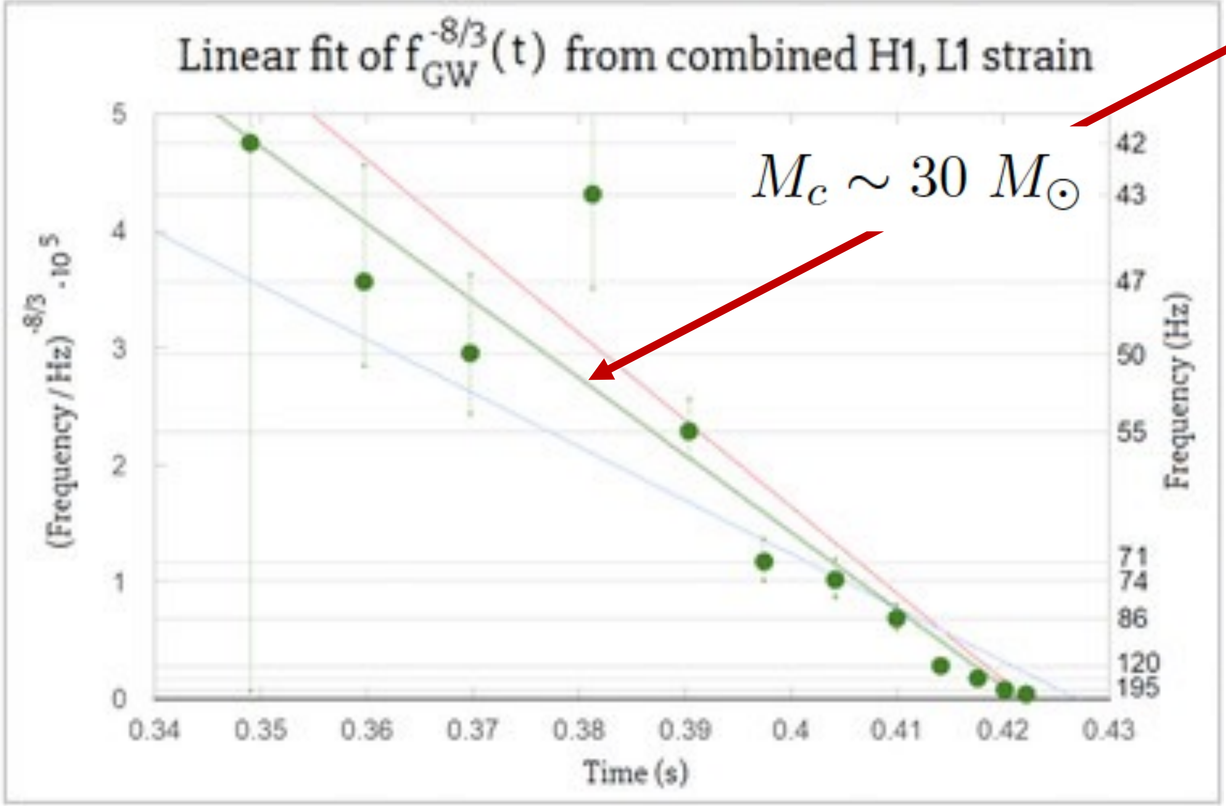
Observation of GW

First **direct detection** of GW (14 Sep 2015): GW150914

$$d_L \sim 45\,\mathrm{Gpc}\left(\frac{1\,\mathrm{Hz}}{f_{\mathrm{gw}}}\right)_{\max}\left(\frac{10^{-21}}{h}\right)_{\max}$$

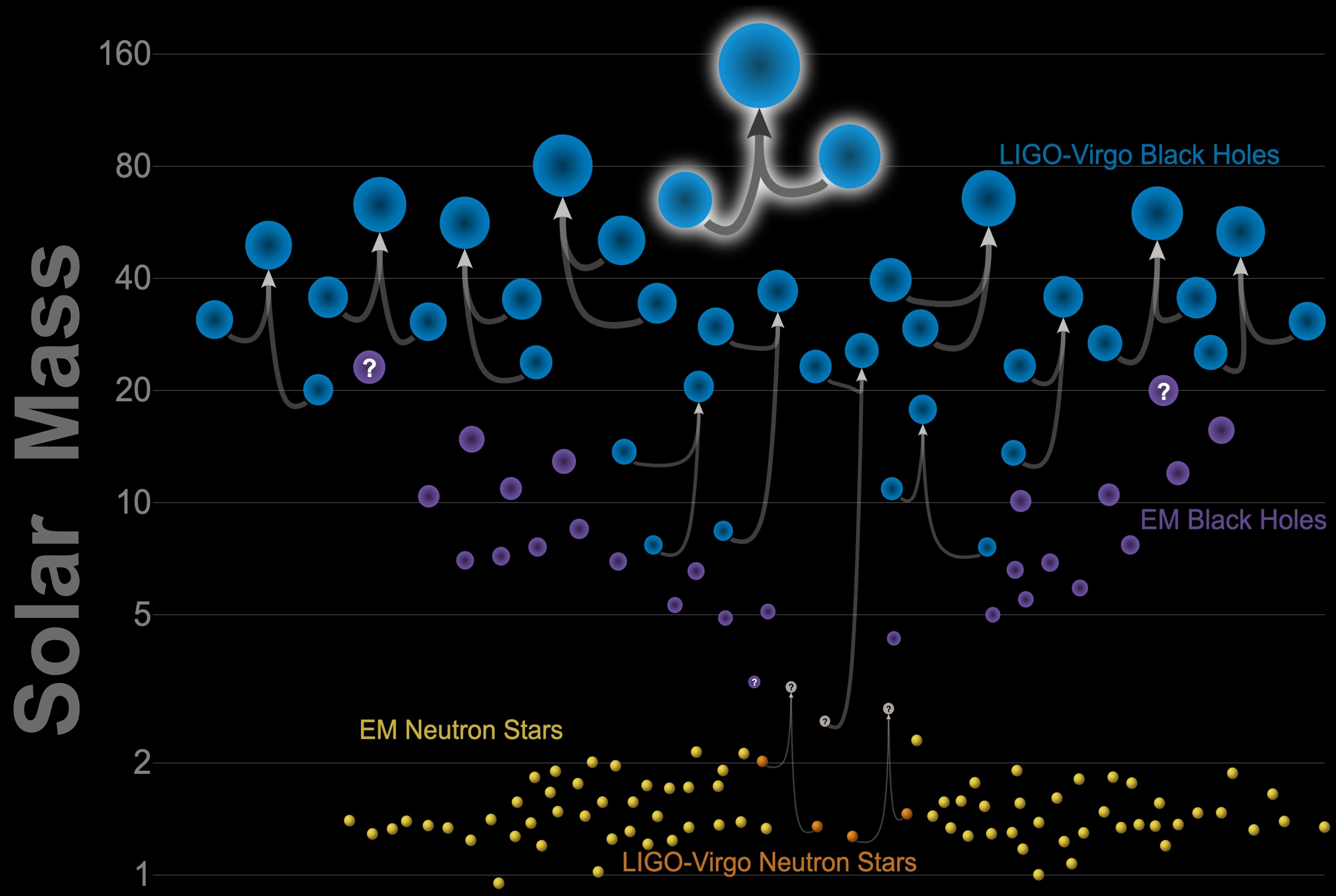
$$f_{\mathrm{gw}}^{-8/3} = \frac{(8\pi)^{8/3}}{5}\left(\frac{GM_c}{c^3}\right)^{5/3}(t_{\mathrm{coal}} - t)$$

$$M_c = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \frac{\dot{f}_{\mathrm{gw}}^{\,3}}{\pi^8 f_{\mathrm{gw}}^{11}}\right)^{1/5}$$



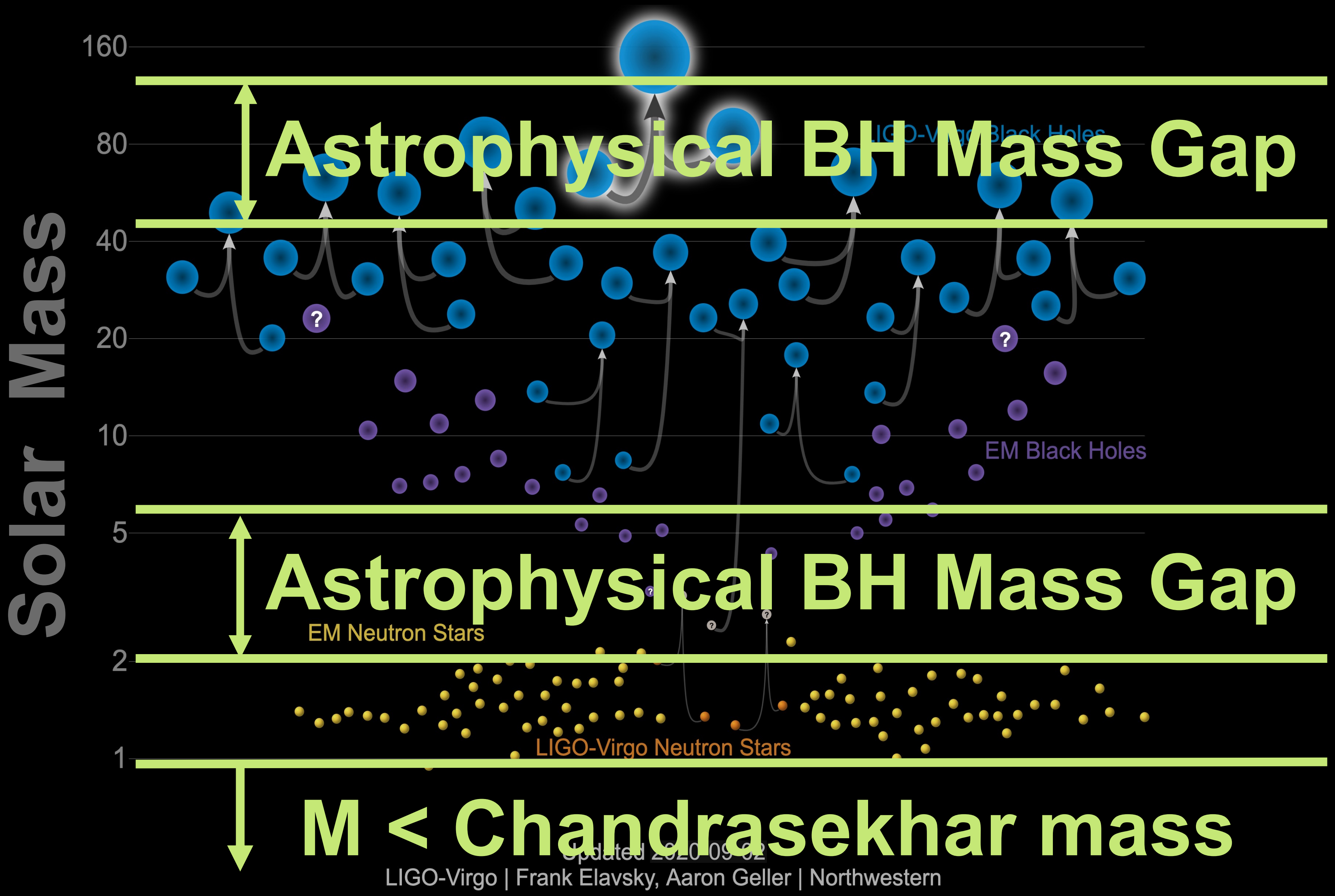
Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

Black Holes and Neutron Stars

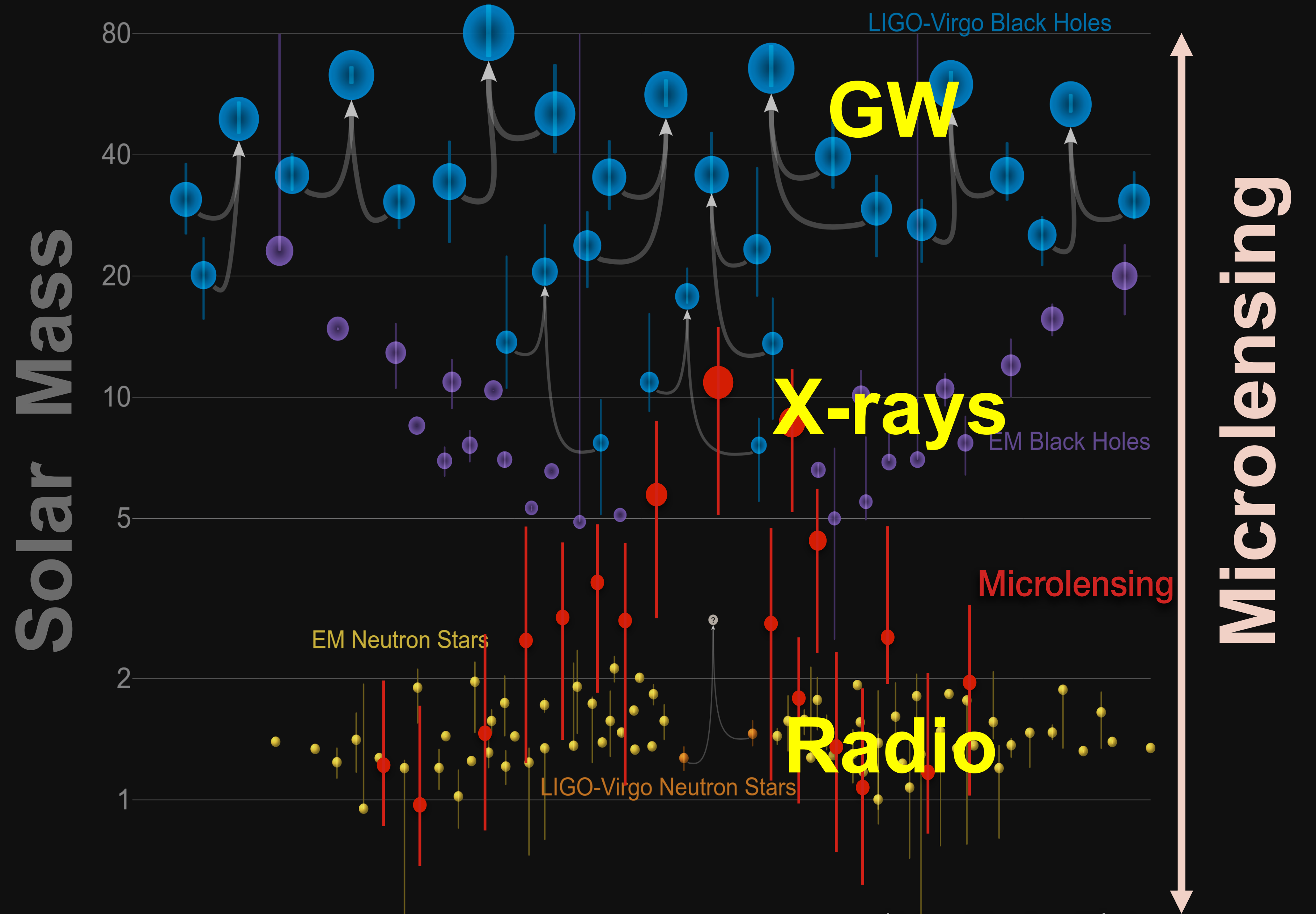


Updated 2020-09-02
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern

Black Holes and Neutron Stars

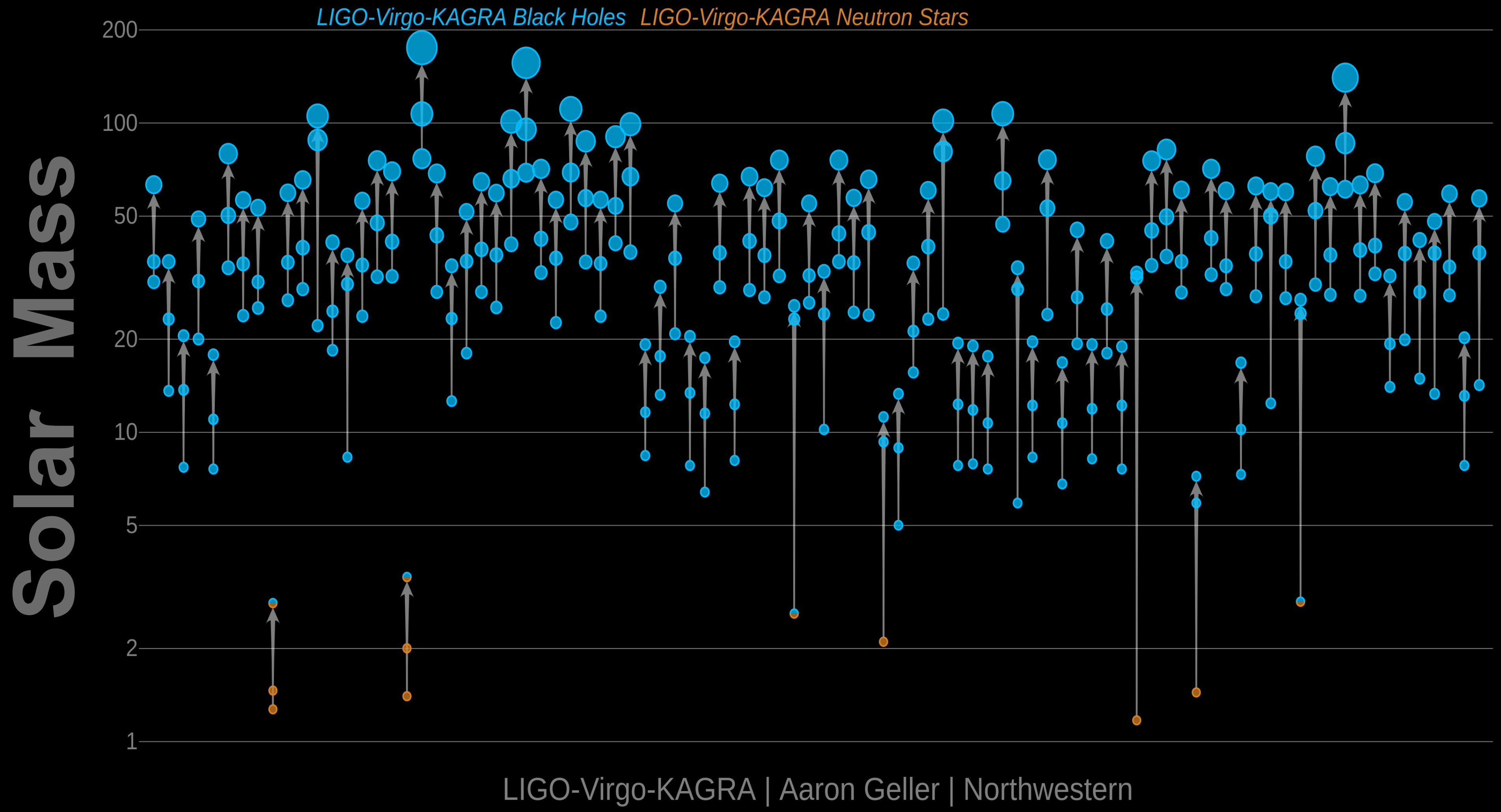


Black Holes and Neutron Stars



GWTC-3

Black Holes and Neutron Stars



GRAVITATIONAL WAVE **MERGER** DETECTIONS

SINCE 2015

OBSERVING RUN

01 2015-2016 | 02 2016-2017 | 03a+b 2019-2020

Date	Primary mass	Secondary mass	Final mass
GW150914	36	31	63
GW151012	23	14	36
GW151226	14	7.7	21
GW170104	31	20	49
GW170608	11	7.6	18
GW170729	50	34	80
GW170809	35	24	56
GW170814	31	25	53
GW170817	1.5	1.3	≤2.8
GW170818	35	27	60
GW170823	40	29	65
GW190403	88	22	105
GW190408	25	18	41
GW190412	30	8.3	37
GW190413	35	24	56
GW190413	48	32	76
GW190421	41	32	70
GW190425	2	1.4	3.2
GW190426	107	77	175
GW190503	43	28	69
GW190512	23	13	35
GW190513	36	18	52
GW190514	39	28	65
GW190517	37	25	59
GW190519	66	41	101
GW190521	95	69	156
GW190521	42	33	71
GW190527	37	23	56
GW190602	69	48	111
GW190620	57	36	87
GW190630	35	24	56
GW190701	54	41	90
GW190706	67	38	99
GW190707	12	8.4	19
GW190708	18	13	30
GW190719	37	21	55
GW190720	13	7.8	20
GW190725	12	6.4	17
GW190727	38	29	64
GW190728	12	8.1	20
GW190731	42	29	67
GW190803	37	27	62
GW190805	48	32	76
GW190814	23	2.6	26
GW190828	32	26	55
GW190828	24	10	33
GW190910	44	36	76
GW190915	35	24	57
GW190916	44	24	66
GW190917	9.3	2.1	11
GW190924	8.9	5	13
GW190925	21	16	35
GW190926	40	23	61
GW190929	81	24	102
GW190930	12	7.8	19
GW191103	12	7.9	19
GW191105	11	7.7	18
GW191109	65	47	107
GW191113	29	5.9	34
GW191126	12	8.3	20
GW191127	53	24	76
GW191129	11	6.7	17
GW191204	27	19	45
GW191204	12	8.2	19
GW191215	25	18	41
GW191216	12	7.7	19
GW191219	31	1.2	32
GW191222	45	35	76
GW191230	49	37	82
GW200105	9	1.9	11
GW200112	36	28	61
GW200115	5.9	1.4	7.2
GW200128	42	33	71
GW200129	34	29	60
GW200202	10	7.3	17
GW200208	38	27	63
GW200208	51	12	61
GW200209	36	27	60
GW200210	24	2.8	27
GW200216	51	30	78
GW200219	38	28	62
GW200220	87	61	141
GW200220	39	28	64
GW200224	40	33	69
GW200225	19	14	32
GW200302	38	20	56
GW200306	28	15	42
GW150914	36	31	63
GW150914	36	31	63
GW150914	36	31	63
GW150914	36	31	63

KEY

BLACK HOLE
PRIMARY MASS
FINAL MASS
NEUTRON STAR (SHOWN AT X10 SCALE)
UNCERTAIN OBJECT
SECONDARY MASS
DATE

31 | 1.2 | 32 | GW191219

UNITS ARE SOLAR MASSES
1 SOLAR MASS = 1.989 x 10^{30}kg

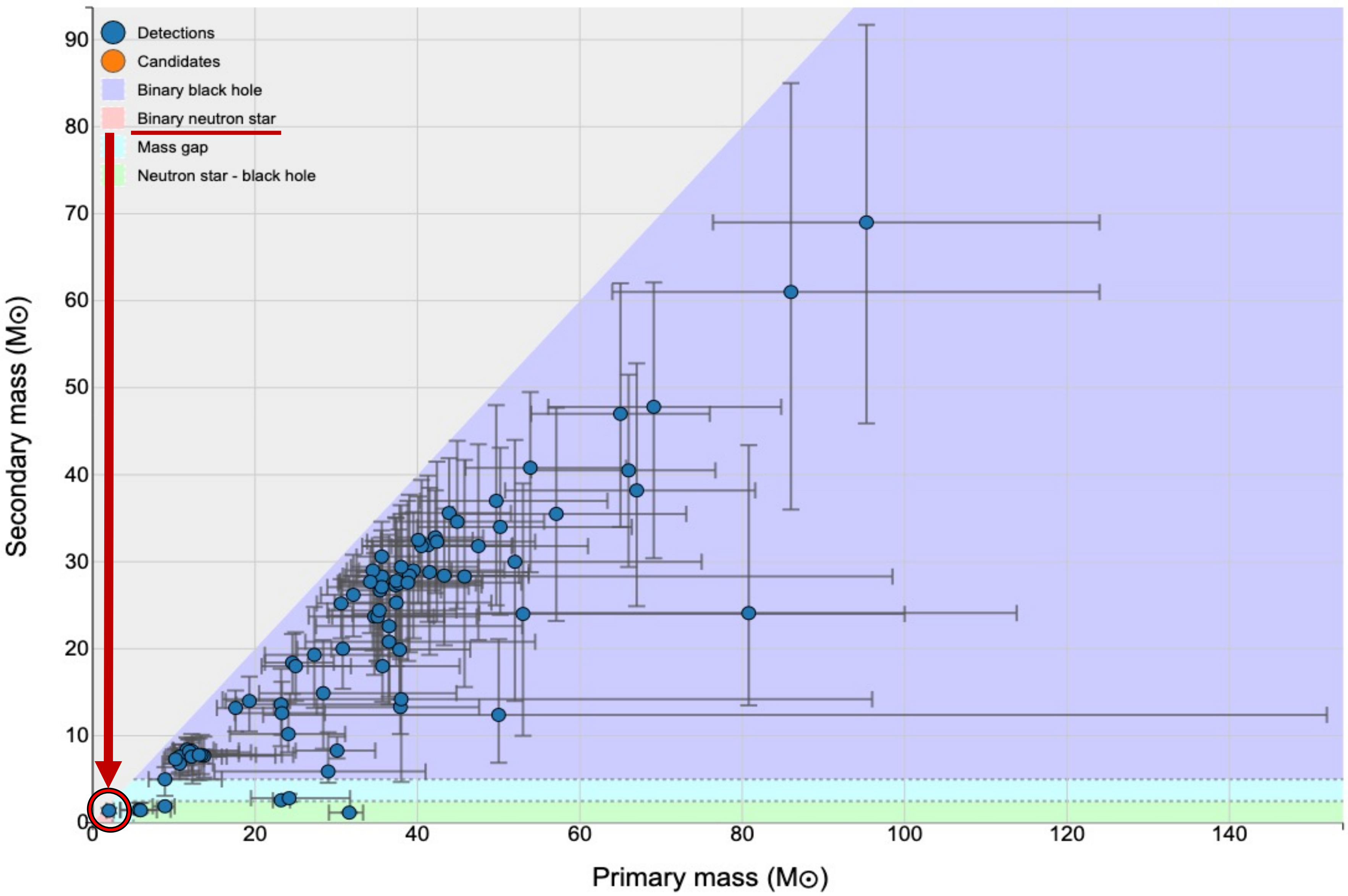
Note that the mass estimates shown here do not include uncertainties, which is why the final mass is sometimes larger than the sum of the primary and secondary masses. In actuality, the final mass is smaller than the primary plus the secondary mass.

The events listed here pass one of two thresholds for detection. They either have a probability of being astrophysical of at least 50%, or they pass a false alarm rate threshold of less than 1 per 3 years.

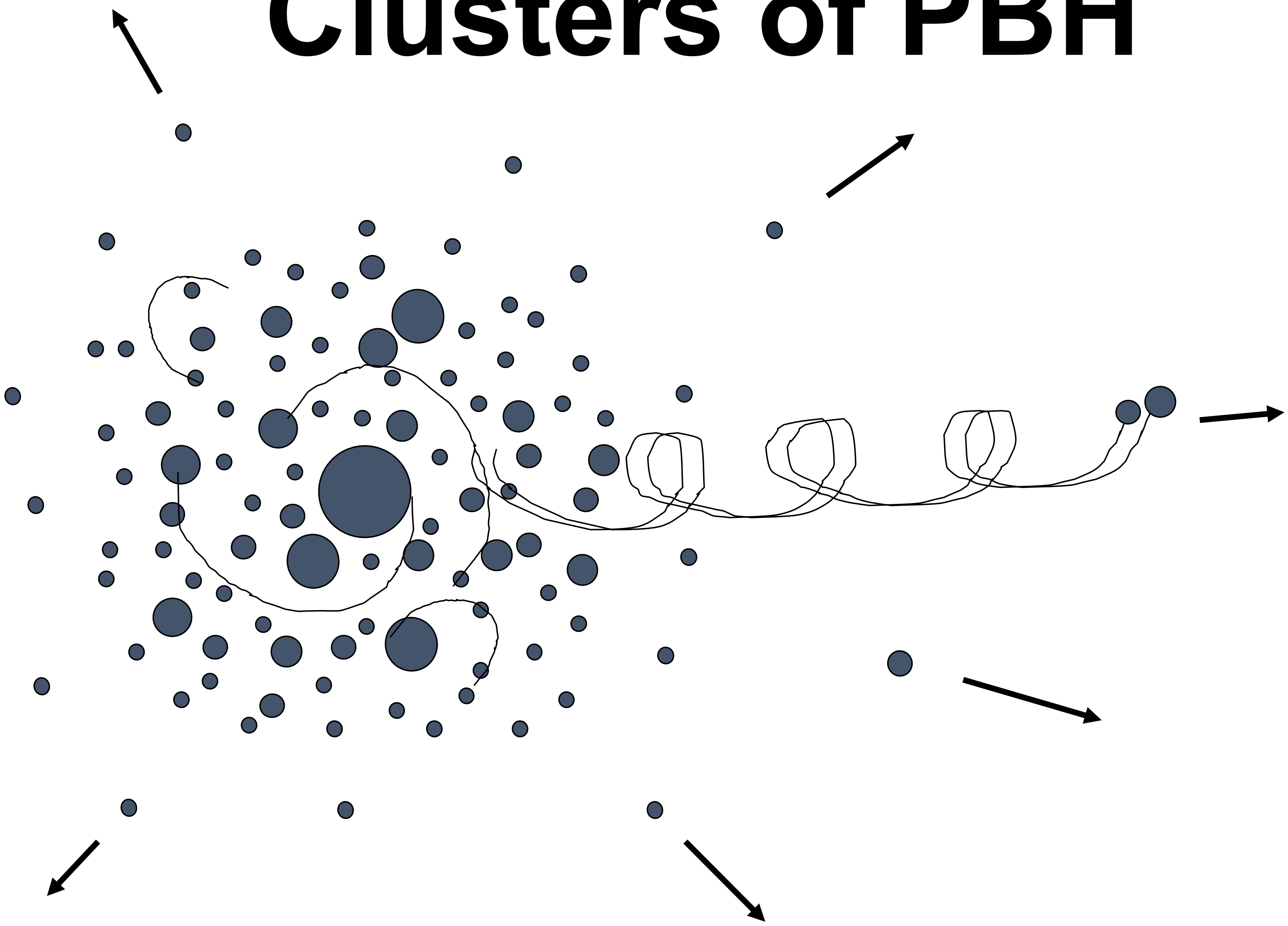
OzGrav
ARC Centre of Excellence for Gravitational Wave Discovery

LIGO VIRGO KAGRA

GWTC-3 (08/11/21)

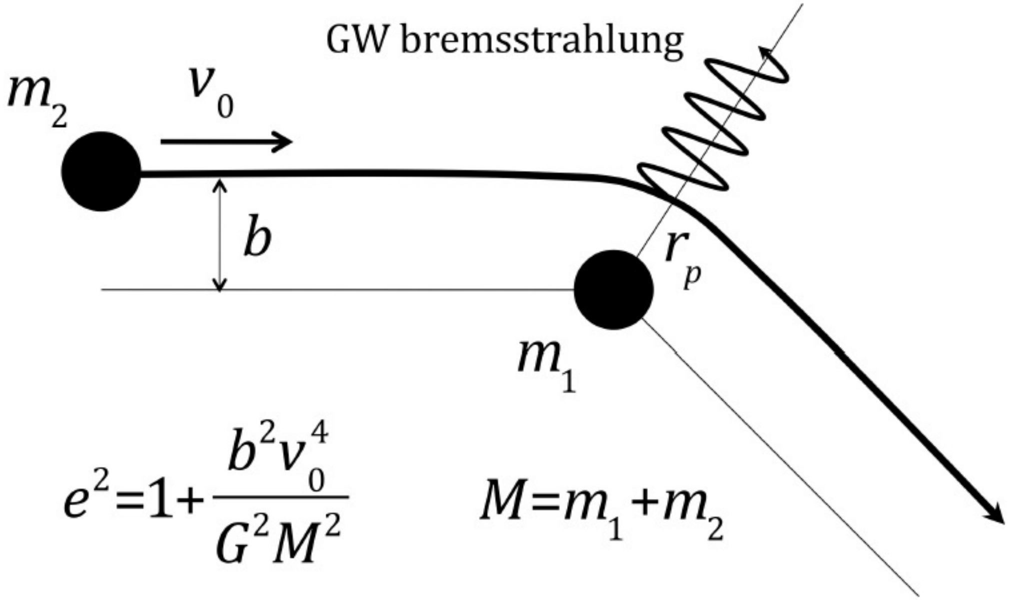


Clusters of PBH



Close Hyperbolic Encounters

Primordial Black Holes in dense clusters scatter off each other



$$e^2 = 1 + \frac{b^2 v_0^4}{G^2 M^2} \qquad M = m_1 + m_2$$

$$\varphi_0 = \arccos\left(-\frac{1}{e}\right)$$

$$r(\varphi) = \frac{b \sin\varphi_0}{\cos(\varphi - \varphi_0) - \cos\varphi_0} = \frac{a\,(e^2 - 1)}{1 + e\cos(\varphi - \varphi_0)}$$

$$r_{\min} = a\,(e-1) = b\sqrt{\frac{e-1}{e+1}} > R_s \equiv \frac{2GM}{c^2}$$

$$b\,v_0 = r_{\min}\,v_{\max} \qquad v_{\max} < c$$

$$\beta \equiv \frac{v_0}{c} < \sqrt{\frac{e-1}{e+1}} \qquad b > R_s\,\frac{(e+1)^{3/2}}{2(e-1)^{1/2}}$$

Close Hyperbolic Encounters

Amplitude and Power emitted in GW

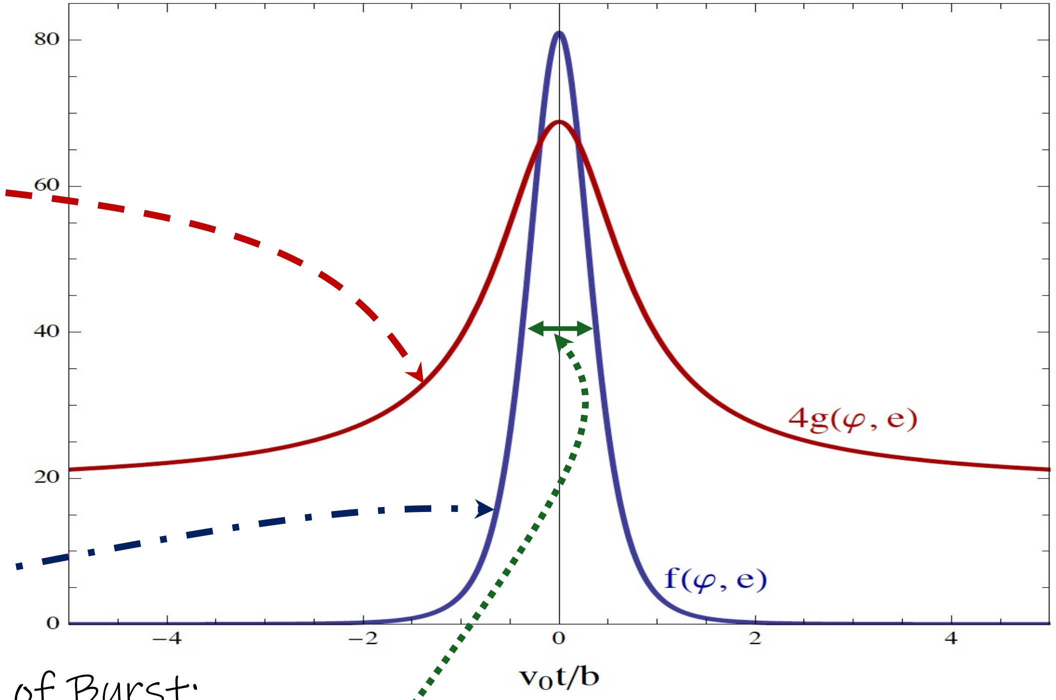
$$Q_{ij} = \mu\, r^2(\varphi)\begin{pmatrix} 3\cos^2\varphi - 1 & 3\cos\varphi\sin\varphi & 0 \\ 3\cos\varphi\sin\varphi & 3\sin^2\varphi - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$h_c = \frac{2G}{R\,c^4}\langle \ddot{Q}_{ij}\ddot{Q}^{ij}\rangle^{1/2}_{i,j=1,2} = \frac{2G\mu\, v_0^2}{R\,c^4}\, g(\varphi,\, e)$$

$$g(\varphi,\, e) = \frac{\sqrt{2}}{e^2-1}\Big[36 + 59\, e^2 + 10\, e^4 + (108 + 47\, e^2)\, e\, \cos(\varphi - \varphi_0) + 59\, e^2\, \cos 2(\varphi - \varphi_0) + 9\, e^3 \cos 3(\varphi - \varphi_0)\Big]^{1/2}$$

$$P = \frac{dE}{dt} = -\frac{G}{45 c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = \frac{32 G\mu^2 v_0^6}{45 c^5\, b^2}\, f(\varphi,\, e)$$

$$f(\varphi,\, e) = \frac{3\,(1 + e\,\cos(\varphi - \varphi_0))^4}{8(e^2-1)^4}\Big[24 + 13\, e^2 + 48\, e\, \cos(\varphi - \varphi_0) + 11\, e^2 \cos 2(\varphi - \varphi_0)\Big]$$



Time duration of Burst:

$$t_{1/2} \simeq 1ms\left(\frac{b}{10^{-8}\mathrm{AU}}\right)\left(\frac{0.01}{\beta}\right)(e-1)\sqrt{\frac{3\ln 2}{e + 35(1+e)e}}$$

Close Hyperbolic Encounters

Power spectrum (frequency domain)

$$\Delta E = \int_{-\infty}^{\infty} P(t)\, dt = \frac{1}{\pi} \int_{0}^{\infty} P(\omega)\, d\omega = -\frac{8}{15} \frac{G^{7/2}}{c^5} \frac{M^{1/2} m_1^2 m_2^2}{r_{min}^{7/2}} f(e)$$

$$\begin{aligned} f(e) \;=\; & \frac{1}{(1+e)^{7/2}} \left[24 \arccos\left(-\frac{1}{e}\right) \left(1 + \frac{73}{24} e^2 + \frac{37}{96}\right) \right. \\ & \left. + \sqrt{e^2 - 1} \left(\frac{301}{6} + \frac{673}{12} e^2 \right) \right] \end{aligned}$$

Quadrupole tensor

$$Q_{ij} = \frac{1}{2} a^2 \mu \begin{pmatrix} (3-e^2)\cosh 2\xi - 8e \cosh \xi & 3\sqrt{e^2-1}(2e \sinh \xi - \sinh 2\xi) & 0 \\ 3\sqrt{e^2-1}(2e \sinh \xi - \sinh 2\xi) & (2e^2 - 3)\cosh 2\xi + 4e \cosh \xi & 0 \\ 0 & 0 & 4e \cosh \xi - e^2 \cosh 2\xi \end{pmatrix}$$

$$r(\xi) \;=\; a(e \cosh \xi - 1) \qquad t(\xi) \;=\; \nu_0(e \sinh \xi - \xi) \qquad \nu_0 = \sqrt{a^3/GM}$$

Close Hyperbolic Encounters

Power spectrum (frequency domain)

$$\begin{aligned} P(\omega) &= \frac{G}{45c^5} \sum_{i,j} |\widehat{\dddot{Q}_{ij}}|^2 \\ &= \frac{G^3\mu^2 M^2}{a^2 c^5} \left(\frac{\pi^2}{180} \, \nu^4 \sum_{i,j} |\widehat{C_{ij}}|^2 \right) \\ &= \frac{G^3\mu^2 M^2}{a^2 c^5} \, \frac{16\pi^2}{180} \, \nu^4 F_e(\nu), \end{aligned}$$

$$\begin{aligned} F_e(\nu) &= \left| \frac{3(e^2-1)}{e} \, H^{(1)\prime}_{i\nu}(i\nu e) + \frac{e^2-3}{e^2} \, \frac{i}{\nu} \, H^{(1)}_{i\nu}(i\nu e) \right|^2 \\ &+ \left| \frac{3(e^2-1)}{e} \, H^{(1)\prime}_{i\nu}(i\nu e) + \frac{2e^2-3}{e^2} \, \frac{i}{\nu} \, H^{(1)}_{i\nu}(i\nu e) \right|^2 \\ &+ \left| \frac{i}{\nu} \, H^{(1)}_{i\nu}(i\nu e) \right|^2 + \frac{18\,(e^2-1)}{e^2} \times \\ &\times \left| \frac{(e^2-1)}{e} \, i \, H^{(1)}_{i\nu}(i\nu e) + \frac{1}{\nu} \, H^{(1)\prime}_{i\nu}(i\nu e) \right|^2 \end{aligned}$$

Total energy emitted:

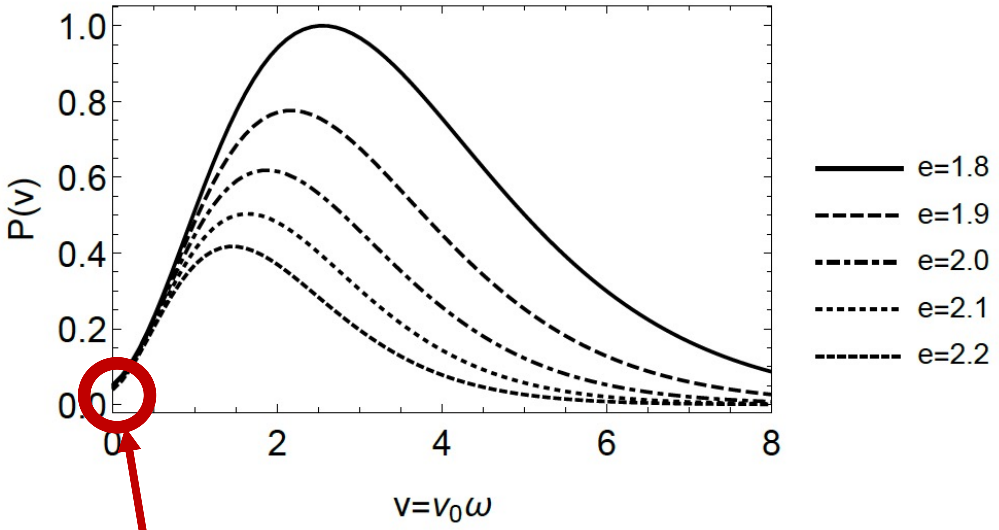
$$\begin{aligned} \Delta E &= \int_{-\infty}^{+\infty} P(t)dt = \int_0^{+\infty} \frac{P(\omega)}{\pi} d\omega \\ &= \left(\frac{G^{7/2}\mu^2 M^{5/2}}{c^5 a^{7/2}} \right) \frac{16\,\pi}{180} \int_0^{+\infty} \nu^4 F_e(\nu) d\nu \,. \end{aligned}$$

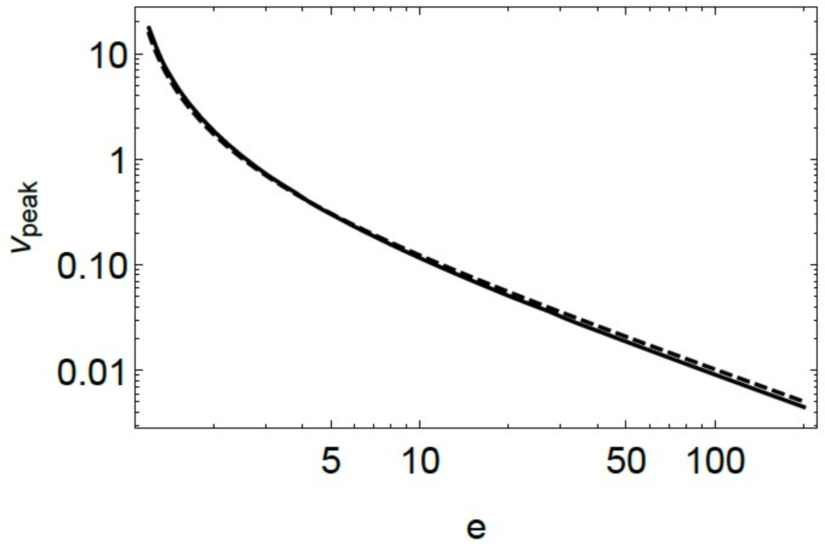
$$\begin{aligned} \nu^4 F_e(\nu) &\simeq \frac{12\, F_y(\nu)}{\pi\, y\, (y^2+1)^2} \, e^{-2\nu z(y)} \\ F_y(\nu) &= \nu \left(1 - y^2 - 3\nu y^3 + 4y^4 + 9\nu y^5 + 6\nu^2 y^6\right) \\ z(y) &= y - \arctan y \,, \qquad y \equiv \sqrt{e^2-1} \end{aligned}$$

Close Hyperbolic Encounters

Peak frequency: $P_{\max} = \frac{32}{45} \frac{q^2 \beta^{10}}{(1+q)^4} \frac{9(e+1)}{(e-1)^5} \frac{c^5}{G}$

$$c^5/G = M_P/t_P = 3.6295 \times 10^{59} \text{ erg/s} = 9.3064 \times 10^{25} \mathcal{L}_\odot;$$





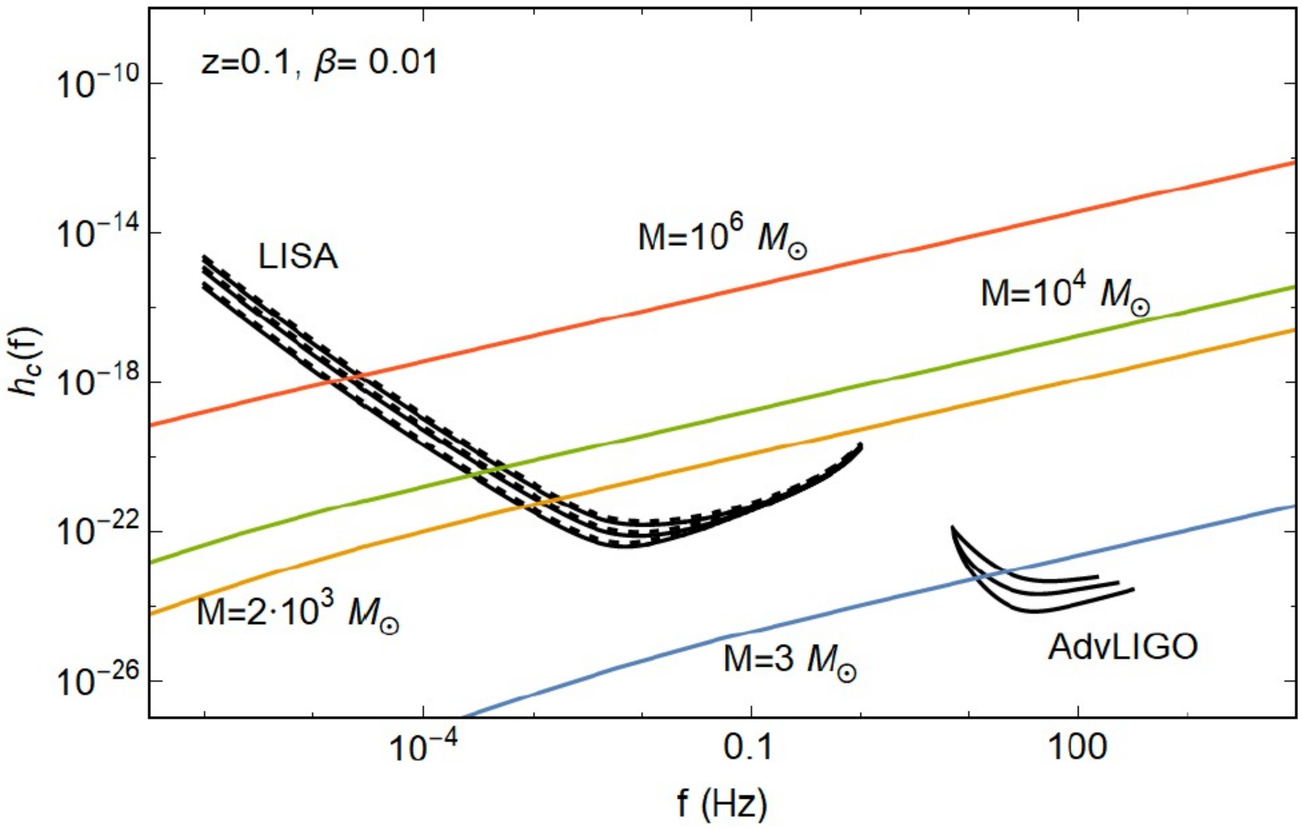
GW memory effect (after scattering, s.t. remembers the event)

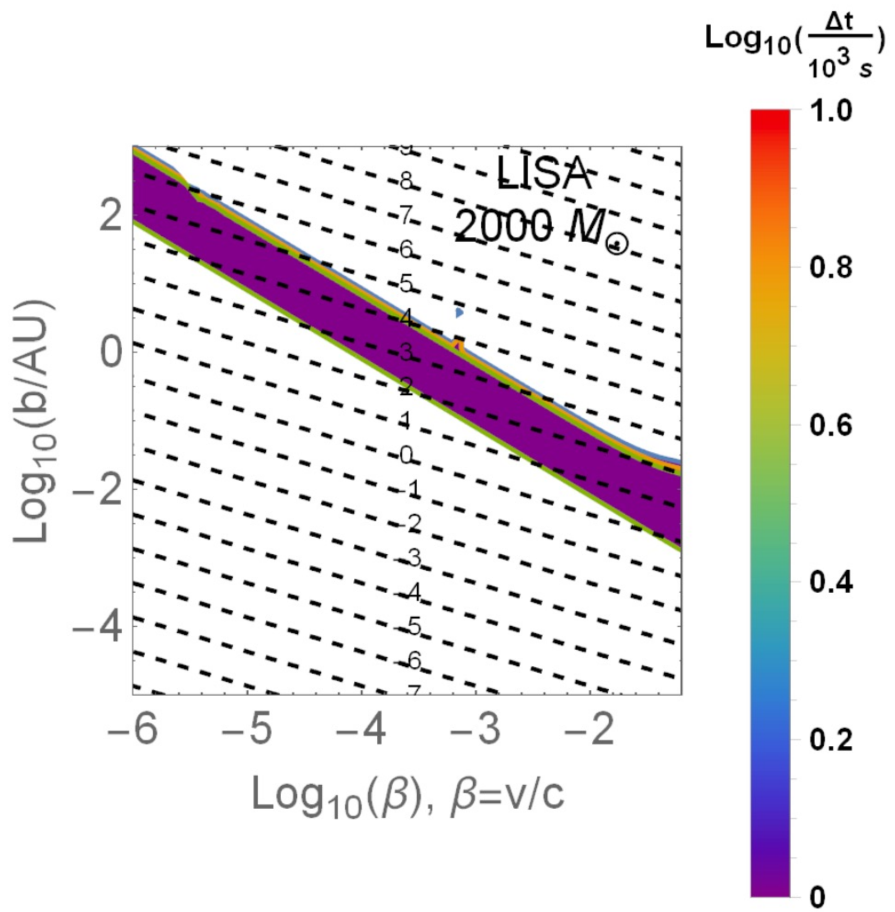
$$P(\omega = 0) = \frac{G^3 \mu^2 M^2}{a^2 c^5} \frac{32\left(e^2-1\right)}{5e^4}.$$

except for $e = 1$ and $e \to \infty$.

Close Hyperbolic Encounters

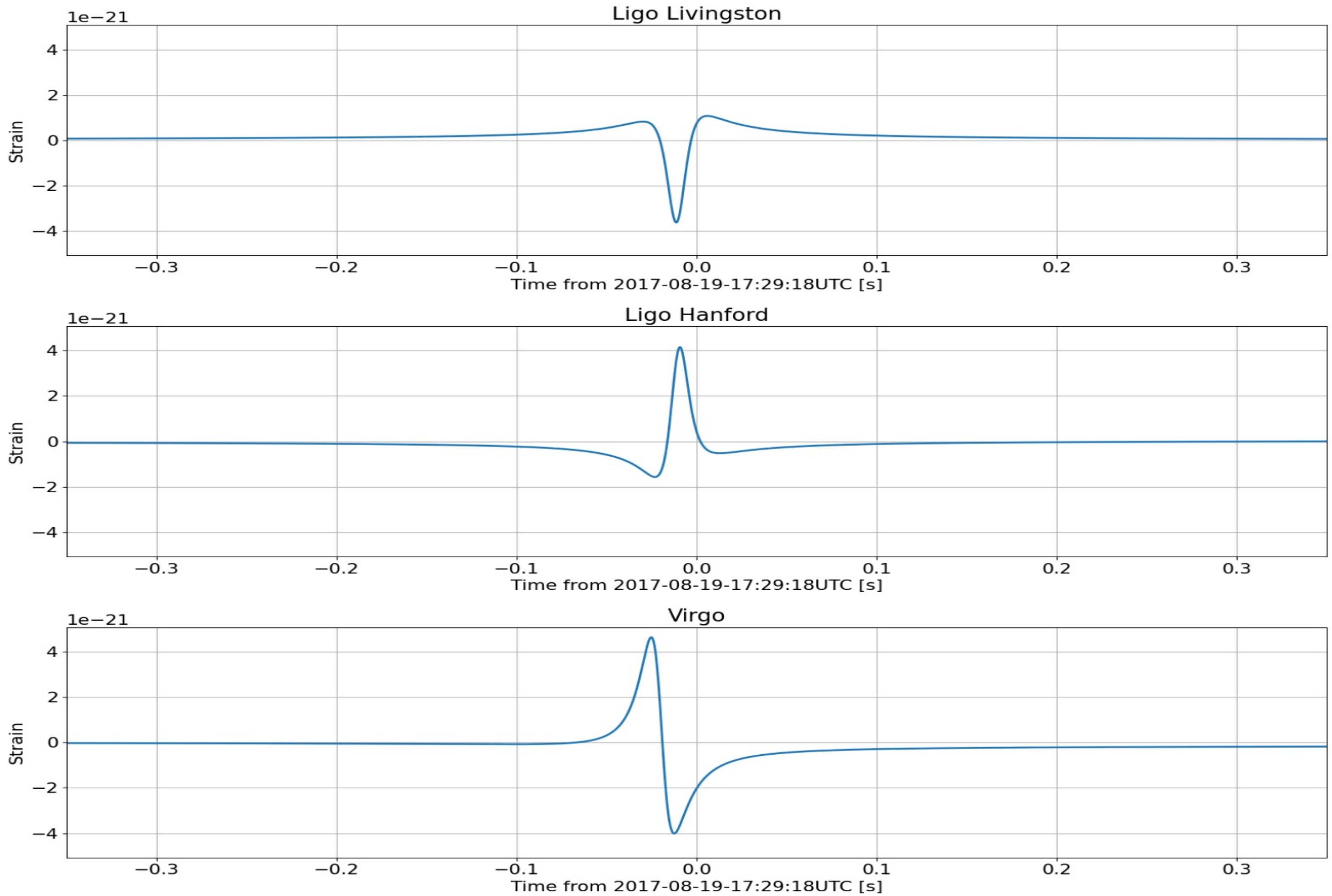
Detection at LIGO/Virgo/KAGRA and LISA





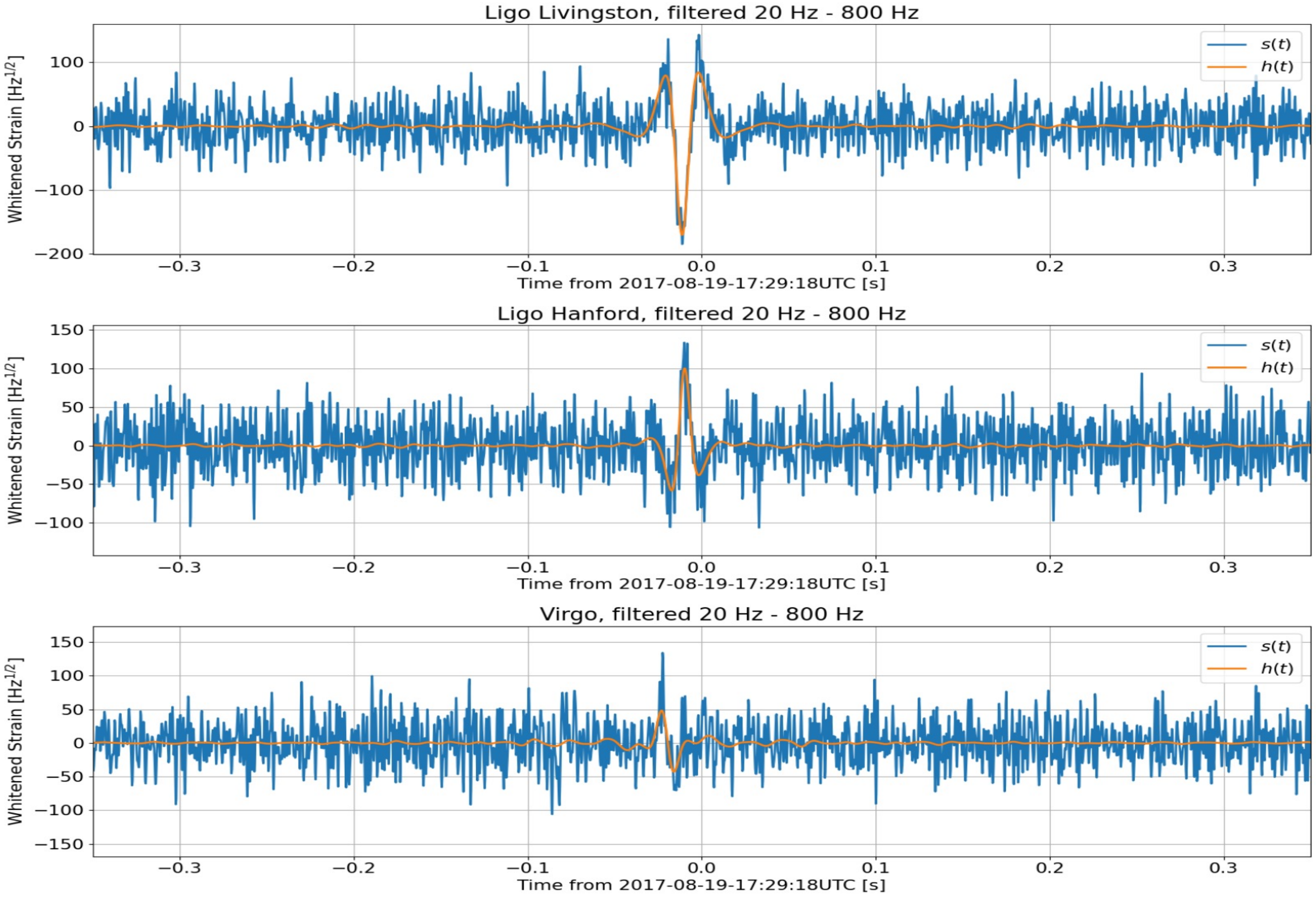
Close Hyperbolic Encounters

How do they look like? Simulated injections



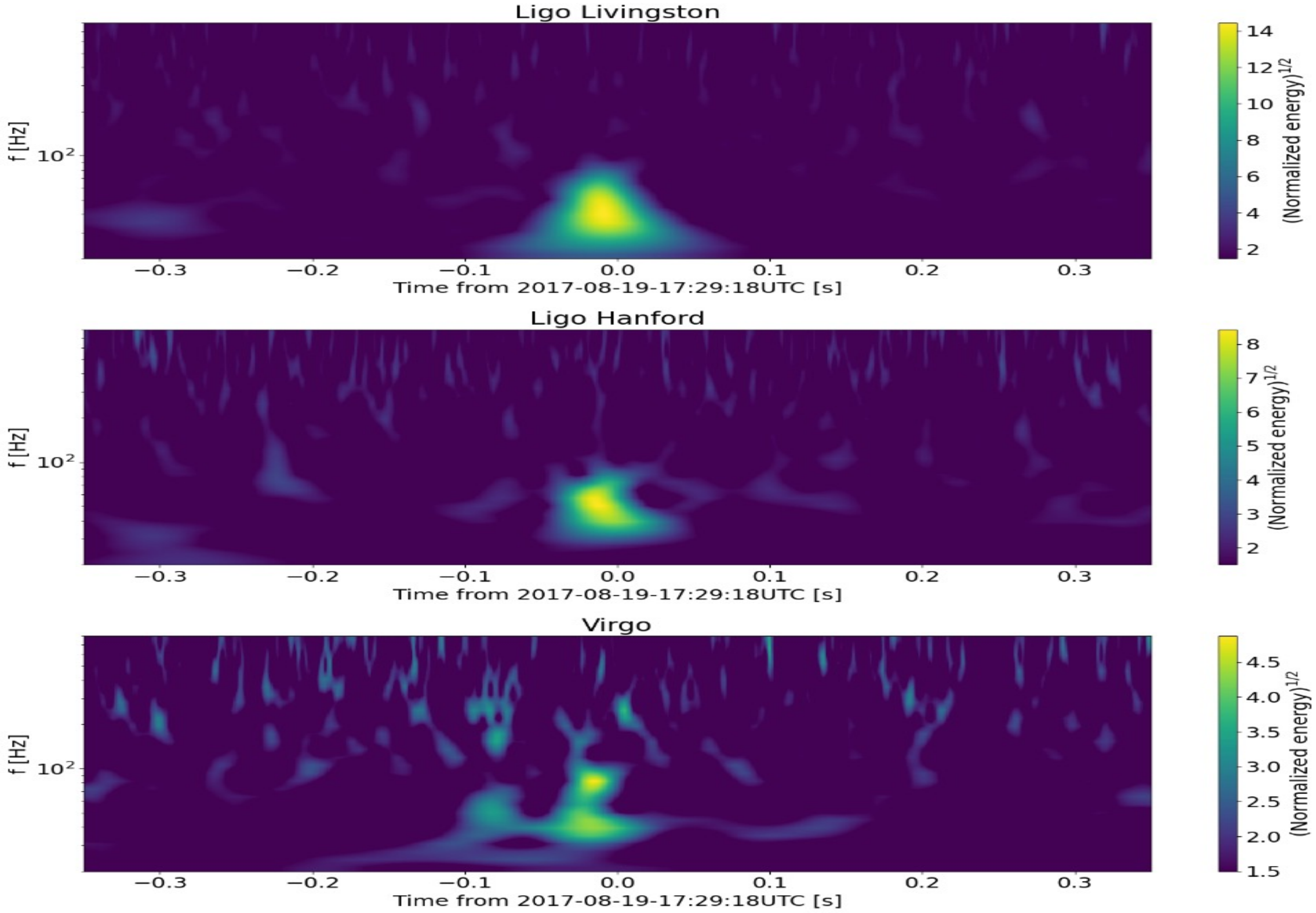
Close Hyperbolic Encounters

How do they look like? Strain amplitude $h(t)$



Close Hyperbolic Encounters

How do they look like? Spectrogram f(t)



Close Hyperbolic Encounters

How do they look like? Can they be confused with glitches?