FUNDAMENTAL PHYSICS WITH GRAVITATIONAL waveS

Winter School on Cosmology

Passo del Tonale

5-10 December 2021

Juan García-Bellido

IFT-UAM/CSIC

Inflationary fluctuations (scalar and tensor)

Quantum to Classical transition

HOMOGENEOUS SCALAR FIELD DYNAMICS

In this subsection I will describe the theoretical basis for the phenomenon of inflation. Consider a scalar field ϕ, a singlet under any given interaction, with an effective potential $V(\phi)$. The Lagrangian for such a field in a curved background is

$$\mathcal{S}_{\rm inf} = \int d^4x\,\sqrt{-g}\,\mathcal{L}_{\rm inf}\,, \qquad \mathcal{L}_{\rm inf} = -\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,, \tag{1}$$

whose evolution equation in a Friedmann-Robertson-Walker metric and for a *homogeneous* field $\phi(t)$ is given by

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0\,, \tag{2}$$

where H is the rate of expansion, together with the Einstein equations,

$$H^2 = \frac{\kappa^2}{3}\Big(\frac{1}{2}\,\dot\phi^2 + V(\phi)\Big)\,, \tag{3}$$

$$\dot H = -\frac{\kappa^2}{2}\,\dot\phi^2\,, \tag{4}$$

where $\kappa^2 \equiv 8\pi G$. The dynamics of inflation can be described as a perfect fluid with a time dependent pressure and energy density given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{5}$$
$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{6}$$

The field evolution equation (2) can then be written as the energy conservation equation,

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{7}$$

If the potential energy density dominates the kinetic energy,

$$V(\phi) \gg \dot{\phi}^2 \quad \Rightarrow \quad p \simeq -\rho \quad \Rightarrow \quad \rho \simeq \text{const.} \quad \Rightarrow \quad H(\phi) \simeq \text{const.}\,,$$

which leads to the solution

$$a(t) \sim \exp(Ht) \quad \Rightarrow \quad \frac{\ddot{a}}{a} > 0 \quad \text{accelerated expansion}\,. \tag{8}$$

Using the definition of the number of e-folds, $N = \ln(a/a_i)$, we see that the scale factor grows exponentially, $a(N) = a_i \exp(N)$. This solution of the Einstein equations solves immediately the flatness problem. Recall that the problem with the radiation and matter eras is that $\Omega = 1$ ($x = 0$) is an unstable critical point in phase-space. However, during inflation, with $p \simeq -\rho \;\Rightarrow\; \omega \simeq -1$, we have that $1 + 3\omega \geq 0$ and therefore $x = 0$ is a stable *attractor* of the equations of motion, see Eq. (**??**). As a consequence, what seemed an *ad hoc* initial condition, becomes a natural *prediction* of inflation. Suppose that during inflation the scale factor increased N e-folds, then

$$x_0 = x_{\rm in}\, e^{-2N}\, \frac{a_{\rm rh}^2 \rho_{\rm rh}}{a_{\rm end}^2 \rho_{\rm end}}\, \frac{T_{\rm rh}^2}{T_{\rm eq}^2}(1+z_{\rm eq}) \;\simeq\; e^{-2N}\, 10^{56} \leq 1 \quad \Rightarrow \quad N \geq 65\,,$$

where we have assumed that inflation ended at the scale $V_{\rm end}$, and the transfer of the inflaton energy density to thermal radiation at reheating occurred almost instantaneously[1] at the temperature $T_{\rm rh} \sim V_{\rm end}^{1/4} \sim 10^{15}$ GeV. Note that we can now have initial conditions with a large uncertainty, $x_{\rm in} \simeq 1$, and still have today $x_0 \simeq 1$, thanks to the inflationary attractor towards $\Omega = 1$. This can be understood very easily by realizing that the three curvature evolves during inflation as

$$^{(3)}R = \frac{6K}{a^2} = \, ^{(3)}R_{\rm in}\, e^{-2N} \quad \longrightarrow \quad 0\,, \quad \text{for} \ \ N \gg 1\,. \tag{9}$$

Therefore, if cosmological inflation lasted over 65 e-folds, as most models predict, then today the universe (or at least our local patch) should be exactly flat, a prediction that has been tested with great (2%) accuracy by WMAP from observations of temperature anisotropies in the microwave background.

[1]There could be a small delay in thermalization, due to the intrinsic inefficiency of reheating, but this does not change significantly the required number of e-folds.

Furthermore, inflation also solves the homogeneity problem in a spectacular way. First of all, due to the superluminal expansion, any inhomogeneity existing prior to inflation will be washed out,

$$\delta_k \sim \left(\frac{k}{aH}\right)^2 \Phi_k \ \propto \ e^{-2N} \quad \longrightarrow \quad 0\,, \quad \text{for} \quad N \gg 1\,. \tag{10}$$

Moreover, since the scale factor grows exponentially, while the horizon distance remains essentially constant, $d_H(t) \simeq H^{-1} = \text{const.}$, any scale within the horizon during inflation will be stretched by the superluminal expansion to enormous distances, in such a way that at photon decoupling all the causally disconnected regions that encompass our present horizon actually come from a single region during inflation, about 65 e-folds before the end. This is the reason why two points separated more than 1° in the sky have the same backbody temperature, as observed by the COBE satellite: they were actually in causal contact during inflation. There is at present no other proposal known that could solve the homogeneity problem without invoquing an acausal mechanism like inflation.

Finally, any relic particle species (relativistic or not) existing prior to inflation will be diluted by the expansion,

$$\rho_{\rm M} \propto a^{-3} \sim e^{-3N} \longrightarrow 0\,, \quad \text{for } N \gg 1\,, \tag{11}$$

$$\rho_{\rm R} \propto a^{-4} \sim e^{-4N} \longrightarrow 0\,, \quad \text{for } N \gg 1\,. \tag{12}$$

Note that the vacuum energy density ρ_v remains constant under the expansion, and therefore, very soon it is the only energy density remaining to drive the expansion of the universe.

THE SLOW-ROLL APPROXIMATION

In order to simplify the evolution equations during inflation, we will consider the slow-roll approximation (SRA). Suppose that, during inflation, the scalar field evolves very slowly down its effective potential, then we can define the slow-roll parameters,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\kappa^2}{2}\frac{\dot{\phi}^2}{H^2} \ll 1\,, \tag{13}$$

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1\,, \tag{14}$$

$$\xi \equiv \frac{\dddot{\phi}}{H^2\dot{\phi}} - \delta^2 \ll 1\,. \tag{15}$$

It is easy to see that the condition

$$\epsilon < 1 \iff \frac{\ddot{a}}{a} > 0 \tag{16}$$

characterizes inflation: it is all you need for superluminal expansion, i.e. for the horizon distance to grow more slowly than the scale factor, in order to solve the homogeneity problem, as well as for the spatial curvature to decay faster than usual, in order to solve the flatness problem.

The number of e-folds during inflation can be written with the help of Eq. (13) as

$$N = \ln \frac{a_{\rm end}}{a_i} = \int_{t_i}^{t_e} H dt = \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon(\phi)}} , \tag{17}$$

which is an exact expression in terms of $\epsilon(\phi)$.

In the limit given by Eqs. (13), the evolution equations (2) and (3) become

$$H^2\left(1-\frac{\epsilon}{3}\right) \simeq H^2 = \frac{\kappa^2}{3}V(\phi)\,, \tag{18}$$

$$3H\dot{\phi}\left(1-\frac{\delta}{3}\right) \simeq 3H\dot{\phi} = -V'(\phi)\,. \tag{19}$$

Note that this corresponds to a reduction of the dimensionality of phase-space from two to one dimensions, $H(\phi,\dot{\phi}) \to H(\phi)$. In fact, it is possible to prove a theorem, for single-field inflation, which states that the slow-roll approximation is an attractor of the equations of motion, and thus we can always evaluate the inflationary trajectory in phase-space within the SRA, therefore reducing the number of initial conditions to just one, the initial value of the scalar field. If $H(\phi)$ only depends on ϕ, then $H'(\phi) = -\kappa^2\dot{\phi}/2$ and we can rewrite the slow-roll parameters (13) as

$$\begin{aligned}
\epsilon &= \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \simeq \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \epsilon_V \ll 1\,, \\
\delta &= \frac{2}{\kappa^2}\frac{H''(\phi)}{H(\phi)} \simeq \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} - \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \eta_V - \epsilon_V \ll 1\,, \\
\xi &= \frac{4}{\kappa^4}\frac{H'(\phi)H'''(\phi)}{H^2(\phi)} \simeq \frac{1}{\kappa^4}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)} - \frac{3}{2\kappa^4}\frac{V''(\phi)}{V(\phi)}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \\
&+ \frac{3}{4\kappa^4}\left(\frac{V'(\phi)}{V(\phi)}\right)^4 \equiv \xi_V - 3\eta_V\epsilon_V + 3\epsilon_V^2 \ll 1\,.
\end{aligned}$$

These expressions define the new slow-roll parameters ϵ_V, η_V and ξ_V. The number of e-folds can also be rewritten in this approximation as

$$N \simeq \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon_V(\phi)}} = \kappa^2 \int_{\phi_i}^{\phi_e} \frac{V(\phi)\, d\phi}{V'(\phi)}\,, \tag{20}$$

a very useful expression for evaluating N for a given effective scalar potential $V(\phi)$.

GAUGE INVARIANT LINEAR PERTURBATION THEORY

The unperturbed (background) FRW metric can be described by a scale factor $a(t)$ and a homogeneous density field $\rho(t)$,

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}\, dx^i dx^j]\,, \tag{21}$$

where η is the conformal time $\eta = \int \frac{dt}{a(t)}$ and the background equations of motion can be written as

$$\mathcal{H}^2 = a^2 H^2 = \frac{\kappa^2}{3} a^2 \rho - K\,, \tag{22}$$

$$\mathcal{H}' - \mathcal{H}^2 = a^2 \dot{H} = K - \frac{\kappa^2}{2} a^2 (\rho + p)\,, \tag{23}$$

where $\mathcal{H} = aH$.

The most general line element, in linear perturbation theory, with both *scalar*, *vector* and *tensor* metric perturbations, is given by

$$ds^2 = a^2(\eta)\Big\{-(1+2\phi)d\eta^2 + 2(B_{|i} - S_i)dx^i d\eta + \Big[(1-2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{(i|j)} + h_{ij}\Big]dx^i dx^j \tag{24}$$

The indices $\{i, j\}$ label the three-dimensional spatial coordinates with metric γ_{ij}, and the $|i$ denotes covariant derivative with respect to that metric. The vector perturbation is transverse $\gamma^{ij}S_{i|j} = \gamma^{ij}F_{i|j} = 0$, and the tensor perturbation h_{ij} corresponds to a symmetric transverse traceless gravitational wave, $\nabla^i h_{ij} = \gamma^{ij}h_{ij} = 0$. In total, these correspond to $n(n+1)/2 = 10$ independent degrees of freedom, 4 scalars, 2 vectors (3 components − 1 transverse condition each) and 2 tensor (6 components − 3 transverse conditions − 1 trace condition).

GENERAL COORDINATE TRANSFORMATIONS

Not all 10 degrees of freedom are physical. As we know, there is a gauge invariance of the theory under general coordinate transformations. The homogeneity of the FRW space-time gives a natural choice of coordinates in the absence of perturbations; however, the presence of first-order perturbations allow a general coordinate (gauge) transformation,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \qquad \begin{cases} \tilde{\eta} = \eta + \xi^0(\eta, x^k)\,, \\ \tilde{x}^i = x^i + \gamma^{ij}\,\xi_{|j}(\eta, x^k) + \xi^i(\eta, x^k)\,, \end{cases} \tag{25}$$

with arbitrary scalar functions ξ^0 and ξ. The vector function ξ^i is a transverse field, $\xi^i_{|i} = 0$. After this gauge transformation, ξ^0 determines the choice of constant$-\eta$ hypersurfaces, while $\xi^{|i}$ and ξ^i select the spatial coordinates within these hypersurfaces. The choice of coordinates is arbitrary to first order and definitions of first-order metric and matter perturbations are thus coordinate (gauge)-dependent. The and matter perturbations are thus coordinate (gauge)-dependent. The result of the gauge transformation (25) acting on any tensor Q is that of the Lie derivative of the background value Q_0 of that physical quantity,

$$\delta\tilde{Q} = \delta Q - \pounds_\xi Q_0 \,. \tag{26}$$

Alternatively, we can obtain the transformed metric components by perturbing the line element,

$$d\eta = d\tilde{\eta} - \xi^{0\prime} d\tilde{\eta} - \xi^0_{|i}\, d\tilde{x}^i \,, \tag{27}$$

$$dx^i = d\tilde{x}^i - \gamma^{ij}(\xi'_{|j} + \xi'_j)\, d\tilde{\eta} - (\xi^{|i}_{\ |j} + \xi^i_{\ |j})\, d\tilde{\xi}^j \,. \tag{28}$$

Substituting them into the line element and using $a(\eta) = a(\tilde{\eta}) - a'(\tilde{\eta})\xi^0$, we get the line element in the new coordinate system, to first order in metric and coordinate transformations,

$$ds^2 = a^2(\tilde{\eta})\Big\{-\Big(1+2(\phi-\mathcal{H}\xi^0-\xi^{0\prime})\Big)d\tilde{\eta}^2+2\Big((B-\xi^0-\xi')_{|i}-(S_i+\xi_i')\Big)d\tilde{\eta}\,d\tilde{x}^i$$
$$+\Big[\Big(1-2(\psi+\mathcal{H}\xi^0)\Big)\gamma_{ij}+2(E-\xi)_{|ij}+2\Big(F_{(i|j)}-\xi_{(i|j)}\Big)+h_{ij}\Big]\,d\tilde{x}^i d\tilde{x}^j\Big\}$$

Since $ds^2 = d\tilde{s}^2$ is invariant under general coordinate transformations, we can read off the transformation equations for the metric perturbations by writing down the new line element with the new metric perturbations as

$$d\tilde{s}^2 = a^2(\tilde{\eta})\Big\{-(1+2\tilde{\phi})d\tilde{\eta}^2+2(\tilde{B}_{|i}-\tilde{S}_i)d\tilde{\eta}d\tilde{x}^i+\Big[(1-2\tilde{\psi})\,\gamma_{ij}+2\tilde{E}_{|ij}+2\tilde{F}_{(i|j)}+\tilde{h}_{ij}\Big]\,d\tilde{x}^i d\tilde{x}^j\Big\} \tag{29}$$

Thus, the gauge transformation of scalar perturbations becomes

$$\tilde{\phi} = \phi - \mathcal{H}\xi^0 - \xi^{0\prime}\,, \qquad \tilde{B} = B + \xi^0 - \xi'\,, \tag{30}$$
$$\tilde{\psi} = \psi + \mathcal{H}\xi^0\,, \qquad \tilde{E} = E - \xi\,, \tag{31}$$

Thus, the gauge transformation of scalar perturbations becomes

$$\tilde{\phi} = \phi - \mathcal{H}\xi^0 - \xi^{0\prime}\,, \qquad \tilde{B} = B + \xi^0 - \xi'\,, \tag{30}$$

$$\tilde{\psi} = \psi + \mathcal{H}\xi^0\,, \qquad \tilde{E} = E - \xi\,, \tag{31}$$

that of vector perturbations is

$$\tilde{S}_i = S_i + \xi_i'\,, \tag{32}$$

$$\tilde{F}_i = F_i - \xi_i\,, \tag{33}$$

and finally, tensor perturbations remain invariant

$$\tilde{h}_{ij} = h_{ij}\,. \tag{34}$$

Alternatively, these metric transformations could have been obtained from the general expression $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ for a coordinate change (25).

SCALAR FIELD PERTURBED EQUATIONS

Consider the action (1) with line element

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Phi)d\mathbf{x}^2\right]$$

in the Longitudinal gauge, where Φ is the gauge-invariant gravitational potential (**??**). Then the gauge-invariant equations for the perturbations on comoving hypersurfaces (constant energy density hypersurfaces) are

$$\begin{aligned}
\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi &= \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2V'(\phi)\delta\phi]\,, \\
-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi &= -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2V'(\phi)\delta\phi]\,, \\
\Phi' + \mathcal{H}\Phi &= \frac{\kappa^2}{2}\phi'\delta\phi\,, \\
\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2V''(\phi)\delta\phi &= 4\phi'\Phi' - 2a^2V'(\phi)\Phi\,.
\end{aligned}$$

This system of equations seem too difficult to solve at first sight. However, there is a gauge invariant combination of Mukhanov variables

$$u \equiv a\delta\phi + z\Phi \,,$$
$$z \equiv a\frac{\phi'}{\mathcal{H}} \,.$$

for which the above equations simplify enormously,

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0 \,,$$
$$\nabla^2\Phi = \frac{\kappa^2}{2}\frac{\mathcal{H}}{a^2}(zu' - z'u) \,, \tag{35}$$
$$\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{\kappa^2}{2}zu \,. \tag{36}$$

From these, we can find a solution $u(z)$, which can be integrated to give $\Phi(z)$, and together allow us to obtain $\delta\phi(z)$.

CANONICAL QUANTIZATION IN PERTURBATION THEORY

Until now we have treated the perturbations as classical, but we should in fact consider the perturbations Φ and $\delta\phi$ as quantum fields. Note that the perturbed action for the scalar mode u can be written as

$$\delta S = \frac{1}{2}\int d^3x\, d\eta \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{37}$$

In order to quantize the field u in the curved background defined by the metric (21), we can write the operator

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[u_k(\eta)\, \hat{a}_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\, \hat{a}_\mathbf{k}^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{38}$$

where the creation and annihilation operators satisfy the commutation relation of bosonic fields, and the scalar field's Fock space is defined through the vacuum condition,

$$[\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'), \tag{39}$$

$$\hat{a}_\mathbf{k}|0\rangle = 0. \tag{40}$$

Note that we are not assuming that the inflaton is a fundamental scalar field, but that is can be written as a quantum field with its commutation relations (as much as a pion can be described as a quantum field at low energies).

If we impose the equal-time commutation relations on the fields themselves,

$$[\hat{u}(\eta, \mathbf{x}), \hat{\Pi}_u^\dagger(\eta, \mathbf{x}')] = i\hbar\delta^3(\mathbf{x} - \mathbf{x}')\,,$$

we find a normalization condition on the modes u_k

$$u_k\, u_k^{*\prime} - u_k'\, u_k^* = i\,, \tag{41}$$

that coincides with the Wronskian of the mode equation,

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0\,. \tag{42}$$

Note that the modes decouple in linear perturbation theory. The ratio $U(\eta) = z''/z$ acts like a time-dependent potential for this 1D Schrödinger like equation (with time $\leftrightarrow$ space),

$$-u_k'' + U(\eta)\, u_k = k^2\, u_k\,.$$

In order to find exact solutions to the mode equation, we will use the slow-roll parameters (13),

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{\kappa^2}{2}\frac{z^2}{a^2}\,, \tag{43}$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}\,, \tag{44}$$

$$\xi = -\left(2 - \epsilon - 3\delta + \delta^2 - \frac{\phi'''}{\mathcal{H}^2\phi'}\right)\,. \tag{45}$$

In terms of these parameters, the conformal time and the effective potential for the u_k mode can be written as

$$\eta = \frac{-1}{\mathcal{H}} + \int \frac{\epsilon da}{a\mathcal{H}} \;\simeq\; \frac{-1}{\mathcal{H}}\frac{1}{1-\epsilon}\,,$$

$$\frac{z''}{z} = \mathcal{H}^2\left[(1+\epsilon-\delta)(2-\delta) + \mathcal{H}^{-1}(\epsilon' - \delta')\right] \;\simeq\; \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\,,$$

Note that the slow-roll parameters, (43) and (44), can be taken as *constant*,[2] to order $\mathcal{O}(\epsilon^2)$,

$$\begin{aligned} \epsilon' &= 2\mathcal{H}\left(\epsilon^2 - \epsilon\delta\right) = \mathcal{O}(\epsilon^2)\,, \\ \delta' &= \mathcal{H}\left(\epsilon\delta - \xi\right) = \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{46}$$

In that case, for constant slow-roll parameters, we can write

$$\eta = \frac{-1}{\mathcal{H}}\frac{1}{1-\epsilon}\,, \tag{47}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right), \qquad \text{where} \qquad \nu = \frac{1+\epsilon-\delta}{1-\epsilon} + \frac{1}{2}\,. \tag{48}$$

[2]For instance, there are models of inflation, like power-law inflation, $a(t) \sim t^p$, where $\epsilon = \delta = 1/p < 1$, that give constant slow-roll parameters.

EXACT SOLUTIONS

We are now going to search for approximate solutions of the mode equation (42), where the effective potential (46) is of order $z''/z \simeq 2\mathcal{H}^2$ in the slow-roll approximation. In quasi-de Sitter there is a characteristic scale given by the (event) horizon size or Hubble scale during inflation, H^{-1}. There will be modes u_k with physical wavelengths much smaller than this scale, $k/a \gg H$, that are well within the de Sitter horizon and therefore do not feel the curvature of space-time. On the other hand, there will be modes with physical wavelengths much greater than the Hubble scale, $k/a \ll H$.

In these two asymptotic regimes, the solutions can be written as

$$u_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \tag{49}$$

$$u_k = C_1(k)\, z \qquad k \ll aH\,. \tag{50}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, u/z becomes constant on superhorizon scales. For approximately constant slow-roll parameters one can find exact solutions to (42), with the effective potential given by (48), that interpolate between the two asymptotic solutions,

exact solution that connects the two regimes

$$u_k(\eta) = \frac{\sqrt{\pi}}{2}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\, (-\eta)^{1/2}\, H_\nu^{(1)}(-k\eta)\,, \tag{51}$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind

e.g. $H_{3/2}^{(1)}(x) = -e^{ix}\sqrt{2/\pi x}(1 + i/x)$,

and ν is given by (48) in terms of the slow-roll parameters. In the limit $k\eta \to 0$, the solution becomes

$$|u_k| = \frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} (-k\eta)^{\frac{1}{2}-\nu} \equiv \frac{C(\nu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu}, \tag{52}$$

$$C(\nu) = 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} (1-\epsilon)^{\nu-\frac{1}{2}} \simeq 1 \qquad \text{for} \quad \epsilon, \delta \ll 1\,. \tag{53}$$

We can now compute Φ and $\delta\phi$ from the super-Hubble-scale mode solution (50), for $k \ll aH$. Substituting into Eq. (36), we find

$$\Phi = C_1 \left(1 - \frac{\mathcal{H}}{a^2} \int a^2 d\eta\right) + C_2 \frac{\mathcal{H}}{a^2}, \tag{54}$$

$$\frac{\delta\phi}{\phi'} = \frac{C_1}{a^2} \int a^2 d\eta - \frac{C_2}{a^2}\,. \tag{55}$$

The term proportional to C_1 corresponds to the growing mode solution, while that proportional to C_2 corresponds to the decaying mode solution, which can soon be ignored. These quantities are gauge invariant but evolve with time outside the horizon, during inflation, and before entering again the horizon during the radiation or matter eras.

We would like to write an expression for a gauge invariant quantity that is also *constant* for superhorizon modes. Fortunately, in the case of adiabatic perturbations, there is such a quantity:

$$\zeta \equiv \Phi + \frac{1}{\epsilon \mathcal{H}} (\Phi' + \mathcal{H}\Phi) = \frac{u}{z} \,, \tag{56}$$

which is constant, see Eq. (50), for $k \ll aH$. In fact, this quantity ζ is identical, for superhorizon modes, to the

ζ is the gauge invariant curvature perturbation $\mathcal{R}_c$

on constant energy density hypersurfaces,

$$\zeta = \mathcal{R}_c + \frac{1}{\epsilon \mathcal{H}^2} \nabla^2 \Phi \,. \tag{57}$$

Using Eq. (35) we can write the evolution equation for $\zeta = \frac{u}{z}$ as

$$\zeta' = \frac{1}{\epsilon\mathcal{H}}\,\nabla^2\Phi \simeq 0$$

which confirms that ζ is constant for (adiabatic) superhorizon modes, $k \ll aH$, but fails for entropy or isocurvature perturbations.

Therefore, we can evaluate the Newtonian potential Φ_k when the perturbation reenters the horizon during radiation/matter eras in terms of the curvature perturbation $\mathcal{R}_k$ when it left the Hubble scale during inflation,

$$\Phi_k = \left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}_k = \frac{3+3\omega}{5+3\omega}\,\mathcal{R}_k = \begin{cases} \frac{2}{3}\,\mathcal{R}_k & \text{radiation era}\,, \\ \frac{3}{5}\,\mathcal{R}_k & \text{matter era}\,. \end{cases} \tag{58}$$

These expressions will be of special importance later.

GRAVITATIONAL WAVE PERTURBATIONS

Let us now compute the tensor or gravitational wave metric perturbations generated during inflation. The perturbed action for the tensor mode can be written as

$$\delta S = \frac{1}{2}\int d^3x\, d\eta\, \frac{a^2}{2\kappa^2}\Big[(h'_{ij})^2 - (\nabla h_{ij})^2\Big]\,, \tag{59}$$

with the tensor field h_{ij} considered as a quantum field,

$$\hat{h}_{ij}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \Big[h_k(\eta)\, e_{ij}(\mathbf{k}, \lambda)\, \hat{a}_{\mathbf{k},\lambda}\, e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.\Big]\,, \tag{60}$$

where $e_{ij}(\mathbf{k}, \lambda)$ are the two polarization tensors,

satisfying symmetric, transverse and traceless conditions

$$e_{ij} = e_{ji}\,, \quad k^i e_{ij} = 0\,, \quad e_{ii} = 0\,, \tag{61}$$

$$e_{ij}(-\mathbf{k}, \lambda) = e^*_{ij}(\mathbf{k}, \lambda)\,, \quad \sum_\lambda e^*_{ij}(\mathbf{k}, \lambda) e^{ij}(\mathbf{k}, \lambda) = 4\,, \tag{62}$$

while the creation and annihilation operators satisfy the usual commutation relation of bosonic fields, Eq. (39). We can now redefine our gauge invariant tensor amplitude as

$$v_k(\eta) = \frac{a}{\sqrt{2}\kappa}\, h_k(\eta)\,, \tag{63}$$

which satisfies the following evolution equation, for each mode $v_k(\eta)$ is decoupled in linear perturbation theory,

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0\,. \tag{64}$$

The ratio a''/a acts like a time-dependent potential for this Schrödinger like equation, analogous to the term z''/z for the scalar metric perturbation. For constant slow-roll parameters, the potential becomes

$$\frac{a''}{a} = 2\mathcal{H}^2\Big(1 - \frac{\epsilon}{2}\Big) = \frac{1}{\eta^2}\Big(\mu^2 - \frac{1}{4}\Big)\,, \tag{65}$$

$$\mu = \frac{1}{1-\epsilon} + \frac{1}{2}\,. \tag{66}$$

We can solve equation (64) in the two asymptotic regimes,

$$v_k = \frac{1}{\sqrt{2k}}\,e^{-ik\eta} \qquad k \gg aH\,, \tag{67}$$

$$v_k = C_3(k)\,a \qquad k \ll aH\,. \tag{68}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, the metric perturbation h_k becomes *constant* on superhorizon scales. For constant slow-roll parameters one can find exact solutions to (64), with effective potential given by (65), that interpolate between the two asymptotic solutions. These are identical to Eq. (51) except the two asymptotic solutions. These are identical to Eq. (51) except for the substitution $\nu \to \mu$,

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\mu+\frac{1}{2})\frac{\pi}{2}} (-\eta)^{1/2} H_\mu^{(1)}(-k\eta) \,, \tag{69}$$

where $H_\mu^{(1)}(z)$ is the Hankel function of the first kind

In the limit $k\eta \to 0$, the solution becomes

$$|v_k| = \frac{C(\mu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\mu} . \tag{70}$$

Since the mode h_k becomes constant on superhorizon scales, we can evaluate the tensor metric perturbation when it reentered during the radiation or matter era directly in terms of its value during inflation.

SCALAR AND TENSOR POWER SPECTRA

Not only do we expect to measure the amplitude of the metric perturbations generated during inflation and responsible for the anisotropies in the CMB and density fluctuations in LSS, but we should also be able to measure its power spectrum, or two-point correlation function in Fourier space. Let us consider first the scalar metric perturbations $\mathcal{R}_k$, which enter the horizon at $a = k/H$. Its correlator is given by

$$\langle 0|\mathcal{R}_k^*\mathcal{R}_{k'}|0\rangle = \frac{|u_k|^2}{z^2}\,\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,,$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}\,\frac{|u_k|^2}{z^2} = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_S^2\left(\frac{k}{aH}\right)^{n_s-1},$$

where we have used

$$\mathcal{R}_k = \zeta_k = \frac{u_k}{z}$$

and Eq. (52). This last equation determines the power spectrum in and Eq. (52). This last equation determines the power spectrum in terms of its amplitude at horizon-crossing,

$$A_S^2 = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2 = \frac{1}{\pi\epsilon}\,\frac{H^2}{M_P^2}$$

, amplitude and tilt,

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}_{\mathcal{R}}(k)}{d\ln k} = 3 - 2\nu = 2\left(\frac{\delta - 2\epsilon}{1-\epsilon}\right) \simeq 2\eta_V - 6\epsilon_V\,, \qquad (71)$$

see Eqs. (20), (20). Note from this equation that it is possible, in principle, to obtain from inflation a scalar tilt which is either positive ($n > 1$) or negative ($n < 1$). Furthermore, depending on the particular inflationary model, we can have significant departures from scale invariance.

Note that at horizon entry $k\eta = -1$, and thus we can alternatively evaluate the tilt as

$$n_s - 1 \equiv -\frac{d\ln \mathcal{P}_\mathcal{R}}{d\ln \eta} = -2\eta\mathcal{H}\left[(1-\epsilon)-(\epsilon-\delta)-1\right] = 2\left(\frac{\delta-2\epsilon}{1-\epsilon}\right) \simeq 2\eta_V - 6\epsilon_V\,, \tag{72}$$

and the running of the tilt

$$\frac{dn_s}{d\ln k} = -\eta\mathcal{H}\left(2\xi + 8\epsilon^2 - 10\epsilon\delta\right) \simeq 2\xi_V + 24\epsilon_V^2 - 16\eta_V\epsilon_V\,, \tag{73}$$

where we have used Eqs. (46).

Let us consider now the tensor (gravitational wave) metric perturbation,

$$h_k = \kappa\sqrt{2}\,\frac{v_k}{a}$$

which enter the horizon at $a = k/H$,

$$\sum_\lambda \langle 0|h^*_{k,\lambda}h_{k',\lambda}|0\rangle = 4\,\frac{2\kappa^2}{a^2}\,|v_k|^2\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,,$$

$$\mathcal{P}_g(k) = 8\kappa^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T},$$

where we have used Eqs. (63) and (70). Therefore, the power spectrum can be approximated by a power-law expression, with amplitude

$$A_T^2 = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2 = \frac{16}{\pi}\,\frac{H^2}{M_P^2}$$

and tilt

$$n_T \equiv \frac{d\ln \mathcal{P}_g(k)}{d\ln k} = 3 - 2\mu = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V < 0\,, \qquad (74)$$

which is always negative. In the slow-roll approximation, $\epsilon \ll 1$, the tensor power spectrum is scale invariant.

Alternatively, we can evaluate the tensor tilt by

$$n_T \equiv -\frac{d\ln \mathcal{P}_g}{d\ln \eta} = -2\eta\mathcal{H}\left[(1-\epsilon) - 1\right] = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V\,, \qquad (75)$$

and its running by

$$\frac{dn_T}{d\ln k} = -\eta\mathcal{H}\left(4\epsilon^2 - 4\epsilon\delta\right) \simeq 8\epsilon_V^2 - 4\eta_V\epsilon_V\,, \qquad (76)$$

where we have used Eqs. (46).

Quantum to Classical

Massless minimally coupled scalar field fluctuations

The fluctuations of a massless minimally-coupled scalar field ϕ during inflation (quasi de Sitter) are quantum fields in a curved background. We will redefine $y(\mathbf{x},t) = a(t)\,\delta\phi(\mathbf{x},t)$, whose action is

$$\mathcal{S} = \int d^4\mathbf{x}\,\mathcal{L}(y,y') = \frac{1}{2}\int d^3\mathbf{x}\,d\eta\,\left[(y')^2 - (\nabla y)^2 + \frac{a''}{a}\,y^2\right]\,, \quad (1)$$

where primes denote derivatives w.r.t. conformal time $\eta = \int dt/a(t) = -1/(aH)$, with H the constant rate of expansion during inflation. Now using the identity $(y')^2 + \frac{a''}{a}\,y^2 = (y' - \frac{a'}{a}\,y)^2 + (\frac{a'}{a}\,y^2)'$, which gives a total derivative in the Lagrangian, we can define the conjugate momentum as $p = \frac{\partial\mathcal{L}}{\partial y'} = y' - \frac{a'}{a}\,y$, and write the corresponding Hamiltonian as

$$\mathcal{H} = p\,y' - \mathcal{L}(y,y') = \frac{1}{2}\left[p^2 + (\nabla y)^2 + 2\,\frac{a'}{a}\,p\,y\right]\,. \quad (2)$$

We can now Fourier transform: $\Phi(\mathbf{k},\eta) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} \Phi(\mathbf{x},\eta)\, e^{-i\,\mathbf{x}\cdot\mathbf{k}}$ all the fields and momenta. Since the scalar field is assumed real, we have: $y(\mathbf{k},\eta) = y^\dagger(-\mathbf{k},\eta)$ and $p(\mathbf{k},\eta) = p^\dagger(-\mathbf{k},\eta)$, and the Hamiltonian becomes

$$\mathcal{H} \;=\; \frac{1}{2}\left[p(\mathbf{k},\eta)\, p^\dagger(\mathbf{k},\eta) + k^2\, y(\mathbf{k},\eta)\, y^\dagger(\mathbf{k},\eta)\right. \tag{3}$$

$$\left. + \frac{a'}{a}\Big(y(\mathbf{k},\eta)\, p^\dagger(\mathbf{k},\eta) + p(\mathbf{k},\eta)\, y^\dagger(\mathbf{k},\eta)\Big)\right] . \tag{4}$$

As we will see later, it is the last term, proportional to a'/a, which is responsible for squeezing.

The Euler-Lagrange equations for this field can be written in terms of the field eigenmodes as a series of uncoupled oscillator equations,

$$\left.\begin{aligned} p' &= -i\,[p, \mathcal{H}] = -k^2\, y - \frac{a'}{a}\, p \\ y' &= -i\,[y, \mathcal{H}] = p + \frac{a'}{a}\, y\,. \end{aligned}\right\} \quad y''(\mathbf{k}, \eta) + \left(k^2 - \frac{a''}{a}\right) y(\mathbf{k}, \eta) = 0\,, \tag{5}$$

where we have used the commutation relation ($\hbar = 1$)

$$\Big[y(\mathbf{k}, \eta), p^\dagger(\mathbf{k}, \eta)\Big] = i\, \delta^3(\mathbf{k} - \mathbf{k}')\,. \tag{6}$$

Heisenberg picture: The field operators

We can now treat each mode as a quantum oscillator, and introduce the corresponding creation and annihilation operators:

$$a(\mathbf{k},\eta) = \sqrt{\frac{k}{2}}\,y(\mathbf{k},\eta) + i\,\frac{1}{\sqrt{2k}}\,p(\mathbf{k},\eta)\,, \tag{7}$$

$$a^{\dagger}(-\mathbf{k},\eta) = \sqrt{\frac{k}{2}}\,y(\mathbf{k},\eta) - i\,\frac{1}{\sqrt{2k}}\,p(\mathbf{k},\eta)\,, \tag{8}$$

which can be inverted to give

$$y(\mathbf{k},\eta) = \frac{1}{\sqrt{2k}}\left[a(\mathbf{k},\eta) + a^{\dagger}(-\mathbf{k},\eta)\right]\,, \tag{9}$$

$$p(\mathbf{k},\eta) = -i\sqrt{\frac{k}{2}}\left[a(\mathbf{k},\eta) - a^{\dagger}(-\mathbf{k},\eta)\right]\,. \tag{10}$$

The usual equal-time commutation relations for fields ($\hbar = 1$ here and throughout),

$$\Big[y(\mathbf{x},\eta),\ p(\mathbf{x}',\eta)\Big] = i\,\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{11}$$

becomes a commutation relation for the creation and annihilation operators,

$$\Big[y(\mathbf{k},\eta),\ p^\dagger(\mathbf{k}',\eta)\Big] = i\,\delta^3(\mathbf{k}-\mathbf{k}') \Rightarrow \Big[a(\mathbf{k},\eta),\ a^\dagger(\mathbf{k}',\eta)\Big] = \delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{12}$$

In terms of these operators, the Hamiltonian becomes:

$$\mathcal{H} \;=\; \frac{1}{2}\Big[k\,\Big(a(\mathbf{k},\eta)\,a^\dagger(\mathbf{k},\eta) + a^\dagger(-\mathbf{k},\eta)\,a(-\mathbf{k},\eta)\Big) \tag{13}$$

$$+i\,\frac{a'}{a}\Big(a^\dagger(-\mathbf{k},\eta)\,a^\dagger(\mathbf{k},\eta) - a(\mathbf{k},\eta)\,a(-\mathbf{k},\eta)\Big)\Big]\,. \tag{14}$$

It is the last (non-diagonal) term which is responsible for squeezing.

The evolution equations, $a' = -i[a, \mathcal{H}]$, can be written as

$$\begin{pmatrix} a'(\mathbf{k}) \\ a^{\dagger\prime}(-\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix} \begin{pmatrix} a(\mathbf{k}) \\ a^{\dagger}(-\mathbf{k}) \end{pmatrix}, \tag{15}$$

whose general solution is, in terms of the initial conditions $a(\mathbf{k}, \eta_0)$,

$$a(\mathbf{k}, \eta) = u_k(\eta)\, a(\mathbf{k}, \eta_0) + v_k(\eta)\, a^{\dagger}(-\mathbf{k}, \eta_0)\,, \tag{16}$$

$$a^{\dagger}(-\mathbf{k}, \eta) = u_k^*(\eta)\, a^{\dagger}(-\mathbf{k}, \eta_0) + v_k^*(\eta)\, a(\mathbf{k}, \eta_0)\,, \tag{17}$$

which correspond to a Bogoliubov transformation of the creation and annihilation operators, and characterizes the time evolution of the system of harmonic oscillators in the Heisenberg representation.

The commutation relation (12) is preserved under the unitary evolution if

$$|u_k(\eta)|^2 - |v_k(\eta)|^2 = 1\,, \tag{18}$$

which gives a normalization condition for these functions.

We can write the quantum fields y and p in terms of these as,

$$y(\mathbf{k},\eta) = f_k(\eta)\, a(\mathbf{k},\eta_0) + f_k^*(\eta)\, a^\dagger(-\mathbf{k},\eta_0)\,, \tag{19}$$

$$p(\mathbf{k},\eta) = -i\,\left[g_k(\eta)\, a(\mathbf{k},\eta_0) - g_k^*(\eta)\, a^\dagger(-\mathbf{k},\eta_0)\right]\,, \tag{20}$$

where the functions

$$f_k(\eta) = \frac{1}{\sqrt{2k}}\left[u_k(\eta) + v_k^*(\eta)\right], \tag{21}$$

$$g_k(\eta) = \sqrt{\frac{k}{2}}\left[u_k(\eta) - v_k^*(\eta)\right], \tag{22}$$

are the field and momentum modes, respectively, satisfying the following equations and initial conditions,

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0\,, \qquad f_k(\eta_0) = \frac{1}{\sqrt{2k}}\,, \tag{23}$$

$$g_k = i\left(f_k' - \frac{a'}{a}\, f_k\right)\,, \qquad g_k(\eta_0) = \sqrt{\frac{k}{2}}\,, \tag{24}$$

as well as the Wronskian condition,

$$i\,(f_k'\, f_k^* - f_k'^*\, f_k) = g_k\, f_k^* + g_k^*\, f_k = 1\,. \tag{25}$$

Squeezing parameters

Since we have two complex functions, f_k and g_k, plus a constraint (25), we can write these in terms of three real functions in the standard parametrization for squeezed states,

$$u_k(\eta) \; = \; e^{-i\,\theta_k(\eta)} \cosh r_k(\eta)\,, \tag{26}$$

$$v_k(\eta) \; = \; e^{i\,\theta_k(\eta)+2i\,\phi_k(\eta)} \sinh r_k(\eta)\,, \tag{27}$$

where r_k is the squeezing parameter, ϕ_k the squeezing angle, and θ_k the phase.

We can also write its relation to the usual Bogoliubov formalism in terms of the functions $\{\alpha_k,\ \beta_k\}$,

$$u_k = \alpha_k\, e^{-ik\eta}\,, \qquad\qquad v_k^* = \beta_k\, e^{ik\eta}\,, \tag{28}$$

which is useful for the adiabatic expansion, and allows one to write the average number of particles and other quantities,

$$n_k \; = \; |\beta_k|^2 = |v_k|^2 = \frac{1}{2k}\left|g_k - k\,f_k\right|^2 = \sinh^2 r_k\,, \tag{29}$$

$$\sigma_k \; = \; 2\mathrm{Re}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Re}\,(u_k^*\,v_k^*) = \cos 2\phi_k\, \sinh 2r_k\,, \tag{30}$$

$$\tau_k \; = \; 2\mathrm{Im}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Im}\,(u_k^*\,v_k^*) = -\sin 2\phi_k\, \sinh 2r_k\,. \tag{31}$$

We can invert these expressions to give $(r_k,\ \theta_k,\ \phi_k)$ as a function of u_k and v_k,

$$\sinh r_k = \sqrt{{\rm Re} v_k^2 + {\rm Im} v_k^2}\,, \qquad \cosh r_k = \sqrt{{\rm Re} u_k^2 + {\rm Im} u_k^2}\,, \quad (32)$$

$$\tan\theta_k = -\frac{{\rm Im} u_k}{{\rm Re} u_k}\,, \qquad \tan 2\phi_k = \frac{{\rm Im} v_k {\rm Re} u_k + {\rm Im} u_k {\rm Re} v_k}{{\rm Re} v_k {\rm Re} u_k - {\rm Im} u_k {\rm Im} v_k}\,. \quad (33)$$

We can now write Eqs. (19) and (20) in terms of the initial values,

$$y(\mathbf{k},\eta) = \sqrt{2k}\, f_{k1}(\eta)\, y(\mathbf{k},\eta_0) - \sqrt{\frac{2}{k}}\, f_{k2}(\eta)\, p(\mathbf{k},\eta_0)\,, \qquad (34)$$

$$p(\mathbf{k},\eta) = \sqrt{\frac{2}{k}}\, g_{k1}(\eta)\, p(\mathbf{k},\eta_0) + \sqrt{2k}\, g_{k2}(\eta)\, y(\mathbf{k},\eta_0)\,, \qquad (35)$$

where subindices 1 and 2 correspond to real and imaginary parts, $f_{k1} \equiv {\rm Re}\, f_k$ and $f_{k2} \equiv {\rm Im}\, f_k$, and similarly for the momentum mode g_k.

The squeezing formalism

Let us now use the squeezing formalism to describe the evolution of the wave function. The equations of motion for the squeezing parameters follow from those of the field and momentum modes,

$$r_k' = \frac{a'}{a}\cos 2\phi_k \,, \tag{36}$$

$$\phi_k' = -k - \frac{a'}{a}\coth 2r_k \,\sin 2\phi_k \,, \tag{37}$$

$$\theta_k' = k + \frac{a'}{a}\tanh 2r_k \,\sin 2\phi_k \,. \tag{38}$$

As we will see, the evolution is driven towards large $r_k \propto N \gg 1$, the number of e-folds during inflation. Thus, in that limit,

$$(\theta_k + \phi_k)' = -\frac{a'}{a}\frac{\sin 2\phi_k}{\sinh 2r_k} \to 0 \,,$$

and therefore $\theta_k + \phi_k \to$ const. We can always choose this constant and therefore $\theta_k + \phi_k \to$ const. We can always choose this constant to be zero, so that the real and imaginary components of the field and momentum modes become

$$f_{k1} = \frac{1}{\sqrt{2k}}\, e^{r_k}\, \cos\phi_k\,, \qquad\qquad f_{k2} = \frac{1}{\sqrt{2k}}\, e^{-r_k}\, \sin\phi_k\,, \tag{39}$$

$$g_{k1} = \sqrt{\frac{k}{2}}\, e^{-r_k}\, \cos\phi_k\,, \qquad\qquad g_{k2} = \sqrt{\frac{k}{2}}\, e^{r_k}\, \sin\phi_k\,. \tag{40}$$

It is clear that, in the limit of large squeezing ($r_k \to \infty$), the field mode f_k becomes purely real, while the momentum mode g_k becomes pure imaginary.

This means that the field (34) and momentum (35) operators become, in that limit,

$$\left.\begin{array}{rcl} \hat{y}(\mathbf{k},\eta) & \to & \sqrt{2k}\, f_{k1}(\eta)\, \hat{y}(\mathbf{k},\eta_0) \\ \hat{p}(\mathbf{k},\eta) & \to & \sqrt{2k}\, g_{k2}(\eta)\, \hat{y}(\mathbf{k},\eta_0) \end{array}\right\} \Rightarrow \hat{p}(\mathbf{k},\eta) \simeq \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\, \hat{y}(\mathbf{k},\eta)\,. \tag{41}$$

As a consequence of this squeezing, information about the initial momentum $\hat{p}_0$ distribution is lost, and the positions (or field amplitudes) at different times commute,

$$\Big[\hat{y}(\mathbf{k},\eta_1),\ \hat{y}(\mathbf{k},\eta_2)\Big] \simeq \frac{1}{2}e^{-2r_k}\cos^2\phi_k \approx 0\,. \tag{42}$$

This result defines what is known as a quantum non-demolition (QND) variable, which means that one can perform succesive measurements of this variable with arbitrary precision without modifying the wave function. Note that $y = a\delta\phi$ is the amplitude of fluctuations produced during inflation, so what we have found is: first, that the amplitude is distributed as a classical Gaussian random field with probability (47); and second that we can measure its amplitude at any time, and as much as we like, without modifying the distribution function.

In a sense, this problem is similar to that of a free non-relativistic quantum particle, described initially by a minimum wave packet, with initial expectation values $\langle x\rangle_0 = x_0$ and $\langle p\rangle_0 = p_0$, which becomes broader by its unitary evolution, and at late times ($t \gg m x_0/p_0$) this Gaussian state becomes an exact WKB state,

$$\Psi(x) = \Omega_R^{-1/2}\,\exp(-\Omega\,x^2/2),$$

with $\mathrm{Im}\Omega \gg \mathrm{Re}\Omega$ (i.e. high squeezing limit). In that limit, $[\hat{x},\ \hat{p}] \approx 0$, and we have lost information about the initial position x_0 (instead of the initial momentum like in the inflationary case), $\hat{x}(t) \simeq \hat{p}(t)\,t/m = p_0\,t/m$ and $\hat{p}(t) = p_0$. Therefore, not only $[\hat{p}(t_1),\ \hat{p}(t_2)] = 0$, but also, at late times, $[\hat{x}(t_1),\ \hat{x}(t_2)] \approx 0$. This explains why we can make subsequent measurements of a particle's position and momentum in a particle physics detector (e.g. a bubble chamber) and still retain all its quantum properties like spin, etc.

The Schrödinger picture: The vacuum wave function

Let us go now from the Heisenger to the Schrödinger picture, and compute the initial state vacuum eigenfunction $\Psi_0(\eta = \eta_0)$. The initial vacuum state $|0, \eta_0\rangle$ is defined through the condition

$$\forall \mathbf{k}\,, \qquad \hat{a}(\mathbf{k}, \eta_0)|0, \eta_0\rangle = \left[\sqrt{\frac{k}{2}}\,\hat{y}_{\mathbf{k}}(\eta_0) + i\,\frac{1}{\sqrt{2k}}\,\hat{p}_{\mathbf{k}}(\eta_0)\right]|0, \eta_0\rangle = 0\,,$$

$$\left[y_k^0 + \frac{1}{k}\,\frac{\partial}{\partial y_k^{0*}}\right]\Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = 0 \;\Rightarrow\; \Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = N_0\, e^{-k\,|y_k^0|^2}$$

where we have used the position representation, $\hat{y}_{\mathbf{k}}(\eta_0) = y_k^0\,,\ \hat{p}_{\mathbf{k}}(\eta_0) = -i\,\frac{\partial}{\partial y_k^{0*}}$, and N_0 gives the corresponding normalization.

We will now study the time evolution of this initial wave function using the unitary evolution operator $S = S(\eta, \eta_0)$, i.e. the state evolves in the Schrödinger picture as $|0, \eta\rangle = S|0, \eta_0\rangle$. Now, inverting (19) and (20)

$$\hat{a}(\mathbf{k}, \eta_0) = g_k^*(\eta)\, \hat{y}(\mathbf{k}, \eta) + i\, f_k^*(\eta)\, \hat{p}(\mathbf{k}, \eta)\,, \tag{43}$$

which, acting on the initial state becomes, $\forall \mathbf{k}\,, \forall \eta$,

$$\begin{aligned} S \left[\hat{y}(\mathbf{k}, \eta) + i\, \frac{f_k^*(\eta)}{g_k^*(\eta)}\, \hat{p}(\mathbf{k}, \eta) \right] S^{-1} S|0, \eta_0\rangle &= 0 \\ \Rightarrow \qquad \left[\hat{y}_{\mathbf{k}}(\eta_0) + i\, \frac{f_k^*(\eta)}{g_k^*(\eta)}\, \hat{p}_{\mathbf{k}}(\eta_0) \right] |0, \eta\rangle &= 0\,, \end{aligned}$$

$$\Rightarrow \qquad \Psi_0 \left(y_{\mathbf{k}}^0, y_{\mathbf{k}}^{0\,*}, \eta \right) = \frac{1}{\sqrt{\pi}\, |f_k(\eta)|}\, e^{-\Omega_k(\eta)\, |y_k^0|^2}\,, \tag{44}$$

where

$$\Omega_k(\eta) \; = \; \frac{g_k^*(\eta)}{f_k^*(\eta)} = k\, \frac{u_k^* - v_k}{u_k^* + v_k} = \frac{1 - 2i\, F_k(\eta)}{2|f_k(\eta)|^2}\,, \tag{45}$$

$$F_k(\eta) \; = \; \mathrm{Im}(f_k^*\, g_k) = \mathrm{Im}(u_k\, v_k) = \frac{1}{2}\, \sin 2\phi_k\, \sinh 2r_k\,. \tag{46}$$

We see that the unitary evolution preserves the Gaussian form of the wave functional. The wave function (44) is called a 2-mode squeezed state.

The normalized probability distribution,

$$P_0\left(y(\mathbf{k},\eta_0), y(-\mathbf{k},\eta_0), \eta\right) = \frac{1}{\pi\,|f_k(\eta)|^2}\,\exp\left(-\frac{|y(\mathbf{k},\eta_0)|^2}{|f_k(\eta)|^2}\right)\,, \qquad (47)$$

is a Gaussian distribution, with dispersion given by $|f_k|^2$.

In fact, we can compute the vacum expectation values,

$$\langle \Delta y(\mathbf{k},\eta)\,\Delta y^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta y^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |f_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (48)$$

$$\langle \Delta p(\mathbf{k},\eta)\,\Delta p^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta p^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |g_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (49)$$

and therefore the Heisenberg uncertainty principle reads

$$\Delta y^2(k)\,\Delta p^2(k) = |f_k|^2\,|g_k|^2 = F_k^2(\eta) + \frac{1}{4} \geq \frac{1}{4}\,. \qquad (50)$$

It is clear that for $\eta = \eta_0$, $\Omega_k(\eta_0) = k$ and $F_k(\eta_0) = 0$, and thus we have initially a minimum wave packet, $\Delta y\,\Delta p = \frac{1}{2}$. However, through its unitary evolution, the function F_k grows exponentially, see (46), and we quickly find $\Delta y\,\Delta p \gg \frac{1}{2}$, corresponding to the semiclassical regime, as we will soon demonstrate rigorously.

The Wigner function

The Wigner function is the best candidate for a probability density of a quantum mechanical system in phase-space. Of course, we know from QM that such a probability distribution function cannot exist, but the Wigner function is just a good approximation to that distribution. Furthermore, for a Gaussian state, this function is in fact positive definite.

Consider a quantum state described by a density matrix $\hat{\rho}$. Then the Wigner function can be written as

$$W(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) = \int\int \frac{dx_1\, dx_2}{(2\pi)^2}\, e^{-i(p_1 x_1 + p_2 x_2)} \left\langle y - \frac{x}{2}, \eta \right| \hat{\rho} \left| y + \frac{x}{2}, \eta \right\rangle .$$

If we substitute for the state our vacuum initial condition $\hat{\rho} = |\Psi_0\rangle\langle\Psi_0|$, with Ψ_0 given by (44), we can perform the integration explicitly to obtain

$$
\begin{aligned}
W_0(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) &= \frac{1}{\pi^2}\exp\left(-\frac{|y|^2}{|f_k|^2} - 4|f_k|^2\left|p - \frac{F_k}{|f_k|^2}\,y\right|^2\right) \\
&\equiv \Phi(y_1,\, p_1)\,\Phi(y_2,\, p_2) \qquad (51) \\
\Phi(y_1,\, p_1) &= \frac{1}{\pi}\exp\left\{-\left(\frac{y_1^2}{|f_k|^2} + 4|f_k|^2\,\bar{p}_1^2\right)\right\}, \qquad (52) \\
\bar{p}_1 &\equiv p_1 - \frac{F_k}{|f_k|^2}\,y_1\,.
\end{aligned}
$$

In general, W_0 describes an asymmetric Gaussian in phase space, whose 2σ contours satisfy

$$
\frac{y_1^2}{|f_k|^2} + 4|f_k|^2\,\bar{p}_1^2 \;\leq\; 1\,. \qquad (53)
$$

For instance, at time $\eta = \eta_0$, we have $y_1^0 = \frac{1}{\sqrt{2k}} = |f_k(\eta_0)|\,,\ p_1^0 = \sqrt{\frac{k}{2}} = 1/2|f_k(\eta_0)|$, and $F_k(\eta_0) = 0$, so that $\bar{p}_1^0 = p_1^0$, and the 2σ

contours become

$$\frac{y_1^2}{y_1^{0\,2}} + \frac{p_1^2}{p_1^{0\,2}} \leq 1\,,$$

which is a circle in phase space.

On the other hand, for time $\eta \gg \eta_0$, we have

$$|f_k| \;\to\; \frac{1}{\sqrt{2k}}\,e^{r_k} \sim y_k^0\,e^N\,, \qquad \text{growing mode}\,, \tag{54}$$

$$\frac{1}{2|f_k|} \;\to\; \sqrt{\frac{k}{2}}\,e^{-r_k} \sim p_k^0\,e^{-N}\,, \qquad \text{decaying mode}\,, \tag{55}$$

so that the ellipse (53) becomes highly “squeezed”.

Note that Liouville’s theorem implies that the volume of phase space is conserved under Hamiltonian (unitary) evolution, so that the area within the ellipse should be conserved. As the probability distribution compresses (squeezes) along the p-direction, it expands along the y-direction. At late times, the Wigner function is highly concentrated direction. At late times, the Wigner function is highly concentrated around the region

$$\bar{p}^2 = \left(p - \frac{F_k}{|f_k|^2}\,y\right)^2 < \frac{1}{4|f_k|^2} \sim e^{-2N} \ll 1\,. \tag{56}$$

We can thus take the above *squeezing limit* in the Wigner function (51) and write the exponential term as a Dirac delta function,

$$W_0(y,p) \overset{r_k\to\infty}{\longrightarrow} \frac{1}{\pi^2}\,\exp\left\{-\frac{|y|^2}{|f_k|^2}\right\}\,\delta\left(p - \frac{F_k}{|f_k|^2}\,y\right)\,. \tag{57}$$

In this limit we have

$$\hat{p}_k(\eta) = \frac{F_k}{|f_k|^2}\,\hat{y}_k(\eta) \;\simeq\; \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\,\hat{y}_k(\eta)\,, \tag{58}$$

so we recover the previous result (41). This explains why we can treat the system as a classical Gaussian random field: the amplitude of the field y is uncertain with probability distribution (47), but once a measurement of y is performed, we can automatically asign to it a *definite* value of the momentum, according to (41).

Note that the condition $F_k^2 \gg 1$ is actually a condition between operators and their commutators/anticommutators. The Heisenberg uncertainty principle states that

$$\Delta_\Psi A \, \Delta_\Psi B \geq \frac{1}{2} \Big| \langle \Psi | [A, \; B] | \Psi \rangle \Big|,$$

for any two hermitian operators (observables) in the Hilbert space of the wave function Ψ. In our case, and in Fourier space, this corresponds to (50)

$$\Delta y^2(k) \, \Delta p^2(k) = F_k^2(\eta) + \frac{1}{4} \geq \frac{1}{4} \Big| \langle \Psi | [y_k(\eta), \; p_k^\dagger(\eta)] | \Psi \rangle \Big|^2, \qquad (59)$$

with $|\Psi\rangle = |0, \eta\rangle$ the evolved wave function.

On the other hand, the phase F_k can be written as

$$F_k = -\frac{i}{2}(g_k\, f_k^* - f_k\, g_k^*) = -\frac{i}{2}\left(\frac{g_k}{f_k}\,|f_k|^2 - |f_k|^2\,\frac{g_k^*}{f_k^*}\right) = \\ = \frac{1}{2}\,\langle\Psi|p(\mathbf{k},\eta)\, y^\dagger(\mathbf{k},\eta) + y(\mathbf{k},\eta)\, p^\dagger(\mathbf{k},\eta)|\Psi\rangle\,, \qquad (60)$$

and we have used that, in the semiclassical limit, we can write $\langle\Psi||y_k(\eta)|^2|\Psi\rangle = |f_k|^2$, as well as $p(\mathbf{k},\eta) = -\,i\,\frac{g_k}{f_k}\,y(\mathbf{k},\eta)$, see (41).

The above relation just indicates that, for any state Ψ, the condition of classicality ($F_k \gg 1$) is satisfied whenever, for that state,

$$\{y_k(\eta),\ p_k^\dagger(\eta)\} \gg |[y_k(\eta),\ p_k^\dagger(\eta)]| = \hbar\,,$$

which is an interesting condition.

Massless scalar field fluctuations on super-horizon scales

The gauge invariant tensor fluctuations (gravitational waves) act as a minimally-coupled massless scalar field during inflation, so we will study here the generation of its fluctuations during quasi de Sitter.

Let us consider here the exact solutions to the equation of motion of a minimally-coupled massless scalar field during inflation or quasi de Sitter, with scale factor $a = -1/H\eta$,

$$f_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right), \tag{61}$$

$$g_k = i\left(f_k' - \frac{a'}{a}\, f_k\right) = \sqrt{\frac{k}{2}}\, e^{-ik\eta}, \tag{62}$$

which satisfy the Wronskian condition, $g_k\, f_k^* + g_k^*\, f_k = 1$. The eigen-modes become which satisfy the Wronskian condition, $g_k\, f_k^* + g_k^*\, f_k = 1$. The eigen-modes become

$$u_k \;=\; e^{-ik\eta}\left(1-\frac{i}{2k\eta}\right) = e^{-ik\eta - i\delta_k}\,\cosh r_k\,, \tag{63}$$

$$v_k \;=\; e^{ik\eta}\,\frac{i}{2k\eta} = e^{ik\eta + i\frac{\pi}{2}}\,\sinh r_k\,, \tag{64}$$

which comparing with (26) and (27) provides the squeezing parameter, the angle and the phase, as inflation proceeds towards $k\eta \to 0^-$,

$$\sinh r_k = \tan\delta_k \;=\; \frac{1}{2k\eta} \to -\infty\,, \tag{65}$$

$$\theta_k = k\eta + \arctan\frac{1}{2k\eta} \to -\frac{\pi}{2}, \qquad \phi_k = \frac{\pi}{4} - \frac{1}{2}\arctan\frac{1}{2k\eta} \to \frac{\pi}{2}, \tag{66}$$

while the imaginary part of the phase of the wave function becomes

$$F_k(\eta) = \frac{1}{2}\,\sin 2\phi_k\,\sinh 2r_k = \frac{1}{2k\eta} \to -\infty\,. \tag{67}$$

The number of scalar field particles produced during inflation grow exponentially, $n_k = |\beta_k|^2 = \sinh^2 r_k = (2k\eta)^{-2} \to \infty$.

Thus, through unitary evolution, the fluctuations will very soon enter the semiclasical regime due to a highly squeezed wave function. The question which remains is when do fluctuations become classical?

Hubble crossing

As we will see, the field fluctuation modes will become semiclassical as their wavelength becomes larger than the only physical scale in the problem, the de Sitter horizon scale, $\lambda_{\rm phys} = 2\pi a/k \gg H^{-1}$.

Therefore, let us consider the general solution to Eq. (23) for the superhorizon modes ($k \ll aH$),

$$f_k(\eta) = C_1(k)\, a + C_2(k)\, a \int^\eta \frac{d\eta'}{a^2(\eta')} = C_1(k)\, a - C_2(k)\, \frac{1}{a^2 H}\,. \quad (68)$$

We can always choose $C_1(k)$ to be real, while $C_2(k)$ will be complex in general. The first term corresponds to the growing mode, while the second term is the decaying mode.

Integrating out g_k from (24), one finds

$$g_k(\eta) = i\,C_2(k)\,\frac{1}{a} - i\,C_1(k)\,k^2\,\frac{1}{a}\int a^2\,d\eta = i\,C_2(k)\,\frac{1}{a} - i\,C_1(k)\,\frac{k^2}{H}\,, \tag{69}$$

where we have added a k^2 term for completeness. To second order in k^2, the Wronskian becomes

$$C_1(k)\,\mathrm{Im}C_2(k)\,\left(1+\frac{k^2}{a^2H^2}\right) \simeq C_1(k)\,\mathrm{Im}C_2(k) = -\frac{1}{2}\,. \tag{70}$$

Comparing with the exact solutions (61), we find, to first order,

$$C_1(k) = \frac{H_k}{\sqrt{2k^3}}\,, \qquad\qquad C_2(k) = -\,\frac{i\,k^{3/2}}{\sqrt{2}\,H_k}\,, \tag{71}$$

where H_k is the Hubble rate at horizon crossing, $k\eta = -1$, i.e. when the perturbation's physical wavelength becomes of the same order as the de Sitter horizon size, $k = aH = \mathcal{H}$.

We are now prepared to answer the question of classicality of the modes. Let us compute the wave function phase shift

$$|F_k| = |\mathrm{Im}(f_k^* \, g_k)| \; = \; \left| C_1^2(k) \, \frac{k^2 a}{H} + |C_2(k)|^2 \, \frac{1}{a^3 H} \right. \tag{72}$$

$$\left. - \; C_1(k) \, \mathrm{Re} C_2(k) \, \left(1 + \frac{k^2}{a^2 H^2}\right) \right| \, . \tag{73}$$

Since only the first term remains after $k\eta \to 0$, we see that $|F_k| \gg 1$ whenever

$$C_1^2(k) = \frac{H_k^2}{2k^3} \gg \frac{H}{k^2 a} \qquad \Rightarrow \qquad \lambda_{\rm phys} = \frac{2\pi a}{k} \gg \lambda_{\rm HC} = \frac{2\pi}{H_k} \, . \tag{74}$$

Therefore, we confirm that modes that start as Minkowski vacuum well inside the de Sitter horizon are stretched by the expansion and become semiclassical soon after horizon crossing, and their amplitude can be described as a classical Gaussian random variable.

Furthermore, the fact that the momentum is immediately defined once the amplitude for a given wavelength is known, implies that there is a fixed temporal phase coherence for all perturbations with the same wavelength. As we know, this implies that inflationary perturbations will induce coherent acoustic oscillations in the plasma just before decoupling, which should be seen in the microwave background anisotropies as acoustic peaks in the angular power spectrum.