

# FUNDAMENTAL PHYSICS WITH GRAVITATIONAL waveS

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# **Second order scalar fluctuations upon reentry**

$$ds^2 = a^2 \left\{ - (1 + 2\Psi) d\eta^2 + \left[ (1 - 2\Psi) \delta_{ij} + \frac{1}{2} h_{ij} \right] dx^i dx^j \right\}$$

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -16 \mathcal{T}_{ij}{}^{\ell m} \left[ \Psi \partial_\ell \partial_m \Psi + 2\partial_\ell \Psi \partial_m \Psi - \partial_\ell \left( \frac{\Psi'}{\mathcal{H}} + \Psi \right) \partial_m \left( \frac{\Psi'}{\mathcal{H}} + \Psi \right) \right]$$

derivatives are taken with respect to conformal time  $\eta$ ,  $\mathcal{H} \equiv a'/a$  is the conformal Hubble the source is evaluated assuming a radiation dominated epoch.

$$\Psi(\eta, \vec{k}) = \frac{2}{3} T(k\eta) \zeta(\vec{k}), \quad \text{where} \quad T(z) = \frac{9}{z^2} \left[ \frac{\sin(z/\sqrt{3})}{z/\sqrt{3}} - \cos(z/\sqrt{3}) \right]$$

One can decompose the tensor field in Fourier space

$$h_{ij}(\eta, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda=R,L} h_\lambda(\eta, \vec{k}) e_{ij,\lambda}(\hat{k}) e^{i\vec{k}\cdot\vec{x}}$$

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<sup>6</sup>The projector  $\mathcal{T}_{ij}{}^{\ell m}$  to the transverse and traceless component is defined in Fourier space in terms of the polarisation tensors  $e_{ij}^\lambda(\vec{k})$  the chiral basis (L, R) as  $\tilde{\mathcal{T}}_{ij}{}^{\ell m}(\vec{k}) = e_{ij}^L(\vec{k}) \otimes e^{L\ell m}(\vec{k}) + e_{ij}^R(\vec{k}) \otimes e^{R\ell m}(\vec{k})$ . The normalisation adopted is  $e_{ij,\lambda}(\vec{k}) e_{ij,\lambda'}^*(\vec{k}) = \delta_{\lambda\lambda'}$ .

the two helicity modes  $h_\lambda$

$$h_\lambda(\eta, \vec{k}) = \frac{4}{9k^3\eta} \int \frac{d^3p}{(2\pi)^3} e_\lambda^*(\vec{k}, \vec{p}) \zeta(\vec{p}) \zeta(\vec{k} - \vec{p}) [\mathcal{I}_c(x, y) \cos(k\eta) + \mathcal{I}_s(x, y) \sin(k\eta)],$$

dimensionless variables  $x = p/k$  and  $y = |\vec{k} - \vec{p}|/k$ , where  $e_\lambda(\vec{k}, \vec{p}) \equiv e_{ij,\lambda}(\hat{k}) \vec{p}_i \vec{p}_j$ ,

$$\mathcal{I}_c(x, y) = -36\pi \frac{(s^2 + d^2 - 2)^2}{(s^2 - d^2)^3} \theta(s - 1),$$

$$\mathcal{I}_s(x, y) = -36 \frac{(s^2 + d^2 - 2)}{(s^2 - d^2)^2} \left[ \frac{(s^2 + d^2 - 2)}{(s^2 - d^2)} \log \frac{(1 - d^2)}{|s^2 - 1|} + 2 \right]$$

$$d \equiv \frac{1}{\sqrt{3}}|x - y|, \quad s \equiv \frac{1}{\sqrt{3}}(x + y), \quad (d, s) \in [0, 1/\sqrt{3}] \times [1/\sqrt{3}, +\infty).$$

The definition of the energy density associated to GWs :

$$\rho_{\text{GW}} = \frac{M_p^2}{4} \left\langle \dot{h}_{ab}(t, \vec{x}) \dot{h}_{ab}(t, \vec{x}) \right\rangle_T$$

brackets denote a time average performed on a timescale  $T$  much greater than the GW phase oscillations ( $T k_i \gg 1$ ) but much smaller than the cosmological time ( $TH \ll 1$ ).

Assuming that the scalar perturbations  $\zeta$  are Gaussian

$$\begin{aligned} \langle \rho_{\text{GW}}(\eta, \vec{x}) \rangle &\equiv \rho_c(\eta) \int d \ln k \Omega_{\text{GW}}(\eta, k) \\ &= \frac{2\pi^4 M_p^2}{81\eta^2 a^2} \int \frac{d^3 k_1 d^3 p_1}{(2\pi)^6} \frac{1}{k_1^4} \frac{\left[ p_1^2 - (\vec{k}_1 \cdot \vec{p}_1)^2/k_1^2 \right]^2}{p_1^3 \left| \vec{k}_1 - \vec{p}_1 \right|^3} \mathcal{P}_\zeta(p_1) \mathcal{P}_\zeta(|\vec{k}_1 - \vec{p}_1|) \left[ \mathcal{I}_c^2(\vec{k}_1, \vec{p}_1) + \mathcal{I}_s^2(\vec{k}_1, \vec{p}_1) \right] \end{aligned}$$

$$\Omega_{\text{GW}}(\eta_0, k) = \frac{a_f^4 \rho_{\text{GW}}(\eta_f, k)}{\rho_r(\eta_0)} \Omega_{r,0} = \frac{g_*(\eta_f)}{g_*(\eta_0)} \left( \frac{g_{*S}(\eta_0)}{g_{*S}(\eta_f)} \right)^{4/3} \Omega_{r,0} \Omega_{\text{GW}}(\eta_f, k),$$

$$f \simeq 8 \text{ mHz} (g_*/10)^{1/4} (T/10^6 \text{ GeV})^{1/4}$$

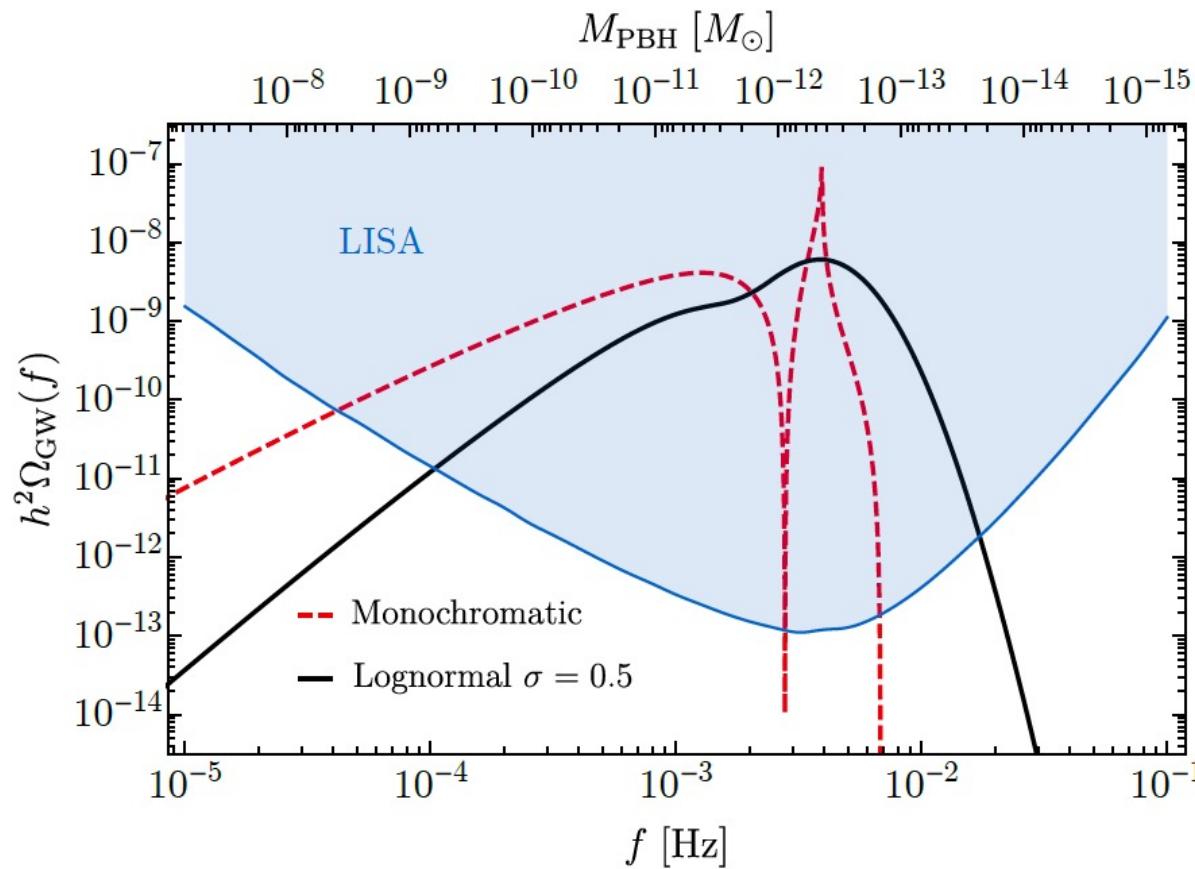
monochromatic power spectrum

$$\mathcal{P}_\zeta(k) = A_s k_\star \delta(k - k_\star)$$

$$\Omega_{\text{GW}}(\eta_0, k) = \frac{\Omega_{r,0} A_s^2}{15552} \frac{g_*(\eta_f)}{g_*(\eta_0)} \left( \frac{g_{*S}(\eta_0)}{g_{*S}(\eta_f)} \right)^{4/3} \left( \frac{4k_\star}{k} - \frac{k}{k_\star} \right)^2 \theta(2k_\star - k) \left[ \mathcal{I}_c^2 \left( \frac{k_\star}{k}, \frac{k_\star}{k} \right) + \mathcal{I}_s^2 \left( \frac{k_\star}{k}, \frac{k_\star}{k} \right) \right]$$

A more realistic spectrum is provided by a lognormal shape with width  $\sigma$

$$\mathcal{P}_\zeta(k) = A_\zeta \exp \left( -\frac{\ln^2(k/k_\star)}{2\sigma^2} \right)$$



$$f \simeq 6 \text{ mHz} \sqrt{\gamma} \left( \frac{M}{10^{-12} M_\odot} \right)^{-1/2}$$

$$A_s = 0.033, A_\zeta = 0.044 \text{ and } \sigma = 0.5$$