Cosmological Simulations

Marco Baldi - Bologna University

XIV Tonale Winter School on Cosmology - 5-10/XII/2021 an introduction to

Cosmological Simulations

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XV Tonale Winter School on Cosmology - 5-10/XII/2021 an introduction to

Cosmological Simulations

(and how to use them to test the Dark Universe)

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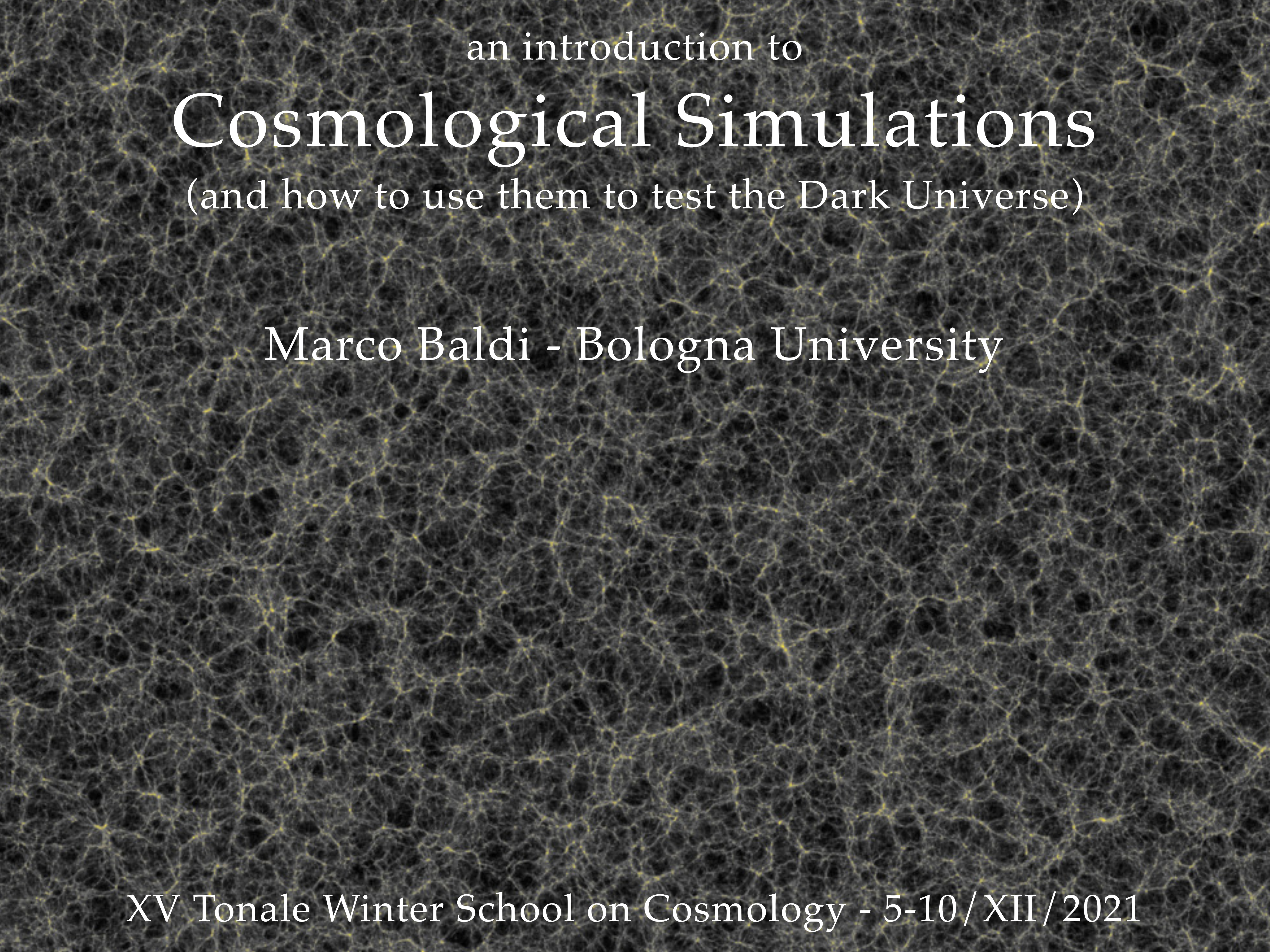

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Two main types of characters in the numerical cosmology business

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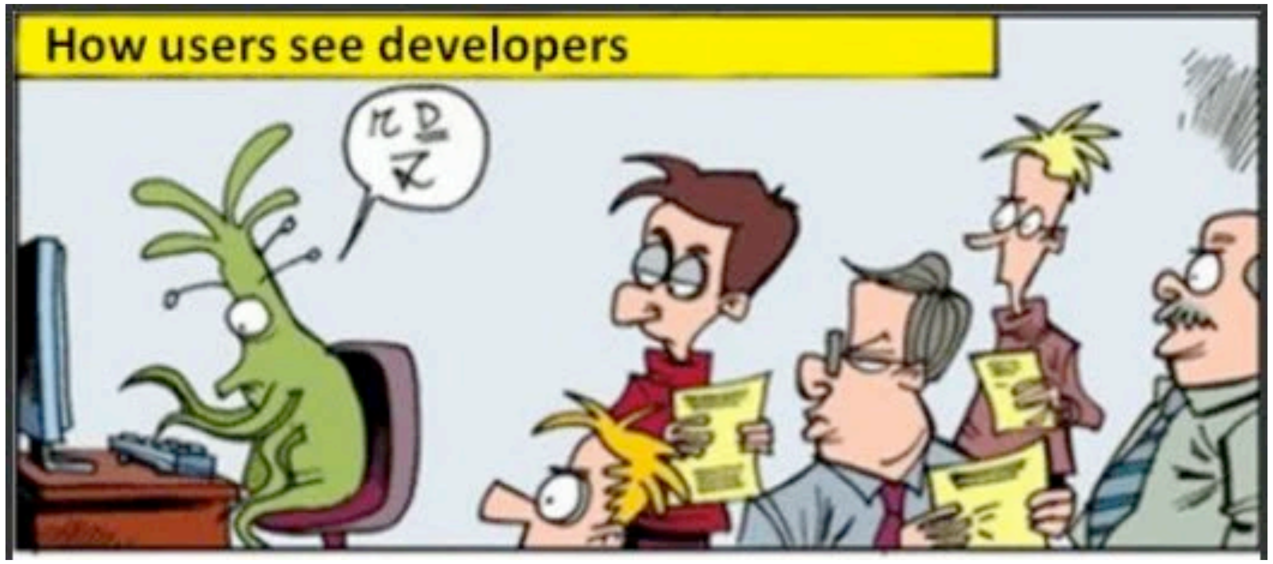

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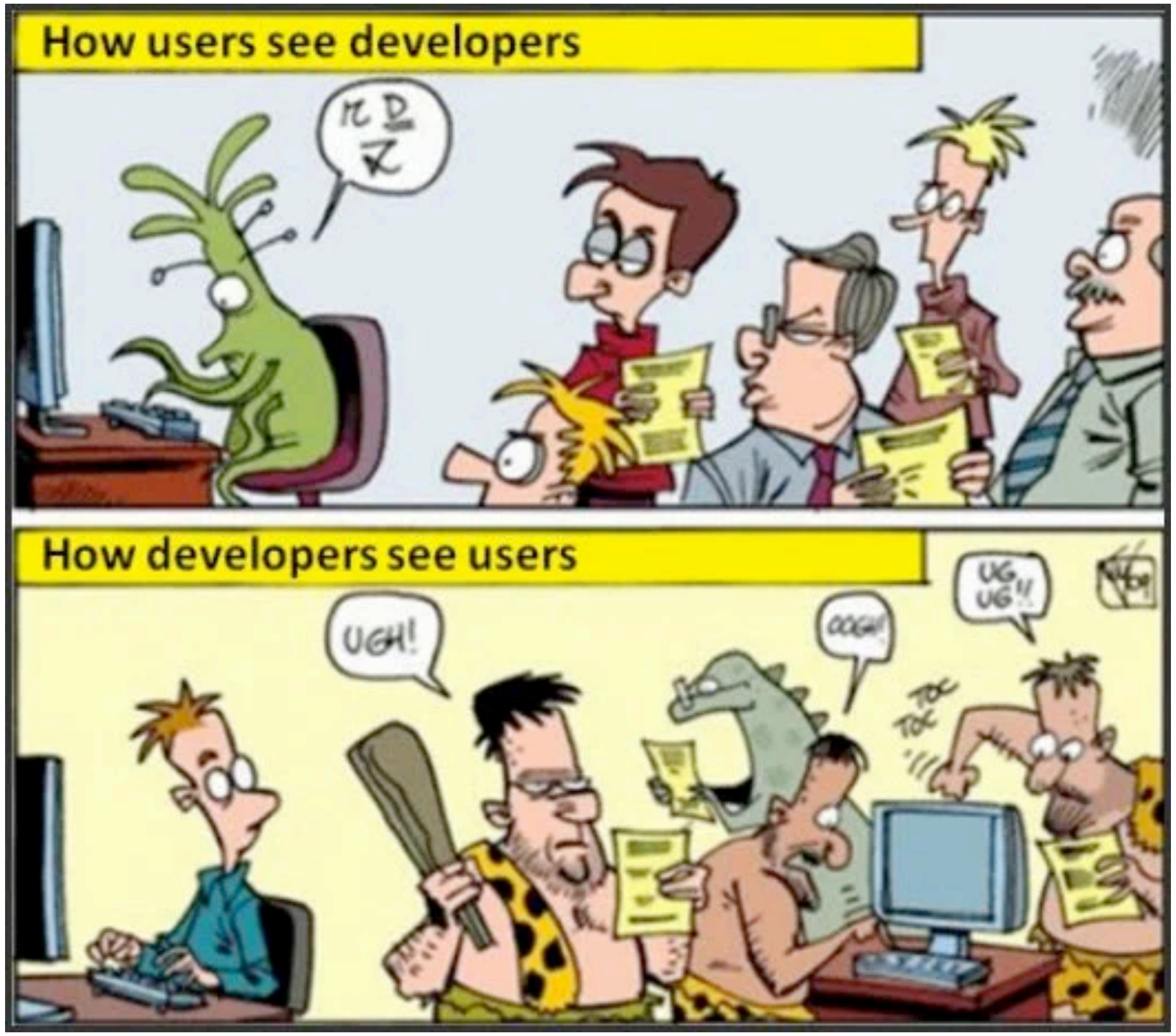

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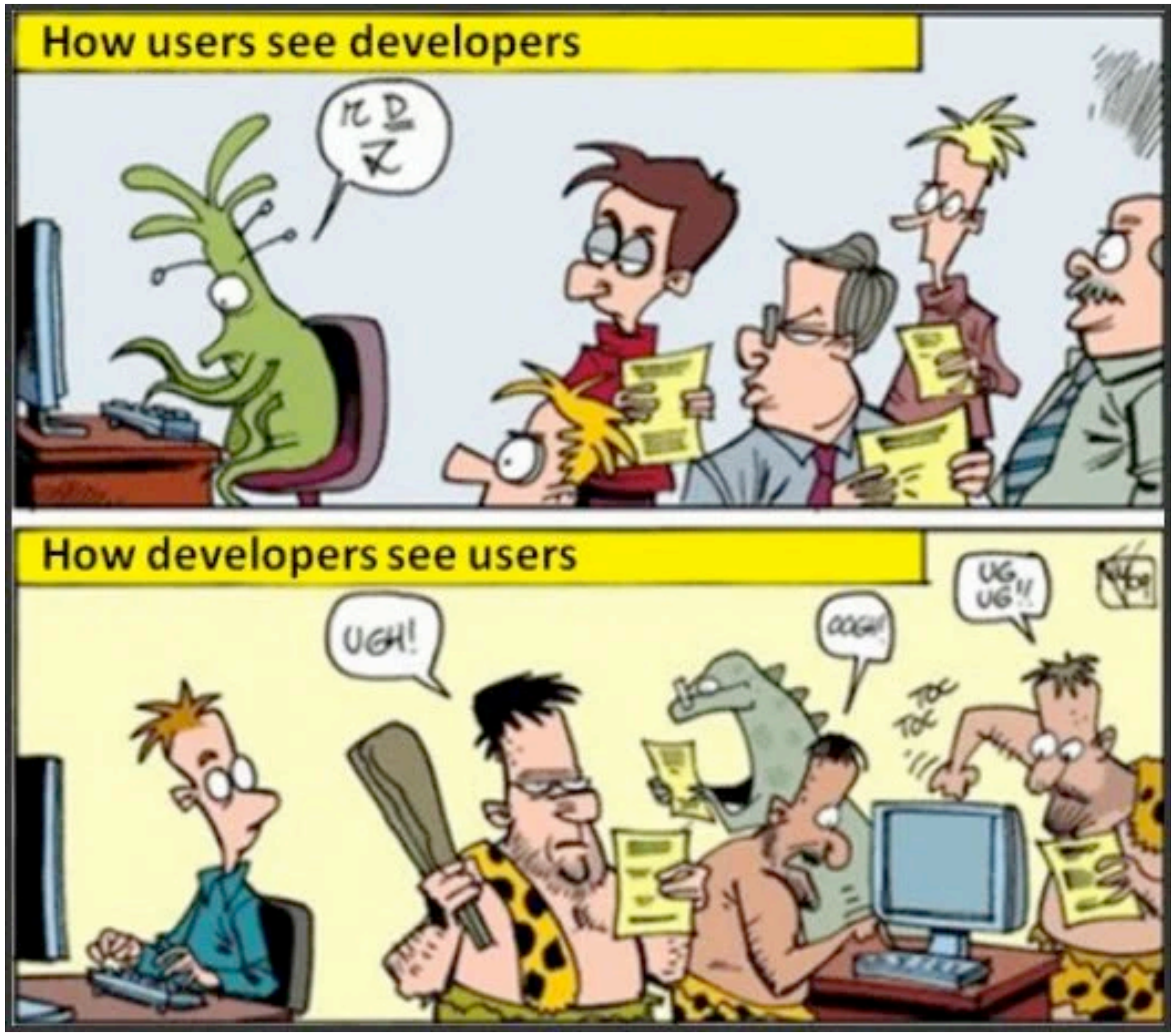

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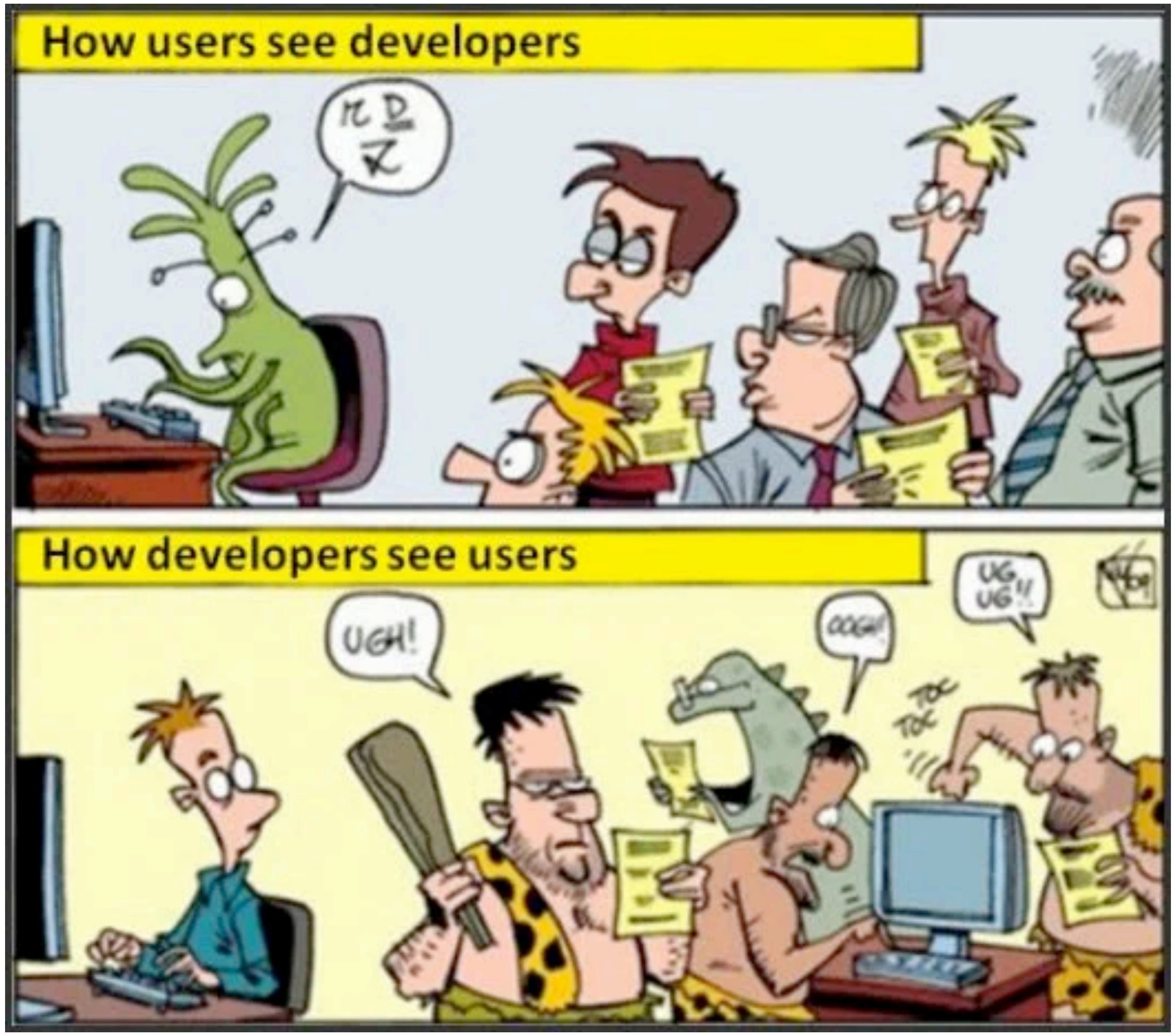

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Outline

Cosmological Simulations: What, Why, and How?

- What are Cosmological Simulations?
- Motivations and applications of creating synthetic universes
- The 3 main steps of cosmological simulations
- The (cosmological) N-body problem: gravitational dynamics of a collisionless system in an expanding space
- Before starting: initial conditions for cosmological simulations
- Solving gravity for a system of N particles: the PM method
- Solving gravity for a system of N particles: the Tree method
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- Time integration

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Cosmological Simulations: What else?

- Beyond LCDM cosmologies: motivations and classification
- Modifying N-body algorithms to include additional physics
- Dark Energy implementations
- Modified Gravity implementations: the example of *f(R)*
- Dark Matter models: massive neutrinos et al.

References

Cosmological Simulations: What, Why, and How?

- *V. Springel, High-Performance Computing and numerical modelling, (2014) arXiv:1412.5187*
- *Kuhlen et al., Numerical Simulations of the Dark Universe: state of the art and next decade (2012) Phys.Dark.Univ.*

Cosmological Simulations: What else?

- *M. Baldi, Dark Energy Simulations (2012), Phys.Dark.Univ.*
- *C. Llinares (2020), Simulations Techniques for Modified Gravity (2018), Int.J.Mod.Phys.D*

What
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simulation [noun]

/ˌsɪmjuˈleɪʃn/
[countable, uncountable] a situation in which a particular set of conditions is created artificially in order to study or experience something that could exist in reality

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… otherwise you fall into one of Borges Paradoxes *(J. L. Borges, "On the exactitude of Science", 1946, inspired by Lewis Carroll's Sylvie and Bruno, 1895)*

What are Cosmological Simulations?

". . . in that Empire, the Cartographer's art achieved such a degree of perfection that the Map of a single Province occupied an entire City, and the Map of the Empire, an entire Province. In time, these vast Maps were no longer sufficient. The Guild of Cartographers created a Map of the Empire, which perfectly coincided with the Empire itself. But Succeeding Generations, with diminished interest in the Study of Cartography, believed that this immense Map was of no use, and not Impiously, they abandoned it to the Inclemency of the Sun and of numerous Winters. In the Deserts of the West ruined Fragments of the Map survive, inhabited by Animals and Beggars; in all the Country there is no other Relic of the Geographical Disciplines."

(*from* Viajes de Varones Prudentes, *Suárez Miranda, book IV, chap. XIV, Lérida, 1658. Quoted by Jorge Luis Borges, Historia universal de la infamia "Etcetera," Buenos Aires, 1935*).

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"What a useful thing a pocket-map is!" I remarked.

"That's another thing we've learned from your Nation," said Mein Herr, "map-making. But we've carried it much further than you. What do you consider the largest map that would be really useful?"

"About six inches to the mile."

"Only six inches!" exclaimed Mein Herr. "We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all ! We actually made a map of the country, on the scale of a mile to the mile!"

"Have you used it much?" I enquired.

"It has never been spread out, yet," said Mein Herr: "the farmers objected: they said it would cover the whole country, and shut out the sunlight ! So we now use the country itself, as its own map, and I assure you it does nearly as well."

from Lewis Carroll, Sylvie and Bruno Concluded, Chapter XI, London, 1895

Why
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Why do we need Cosmological Simulations?

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6. More motivations at the end…

How
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Setting up the stage

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Not an easy task as the Universe is:

- Very **LARGE** —> a fair (i.e. un-biased) replica should encompass a representative portion of the universe, which will contain a large number of degrees of freedom (i.e. the system is very **COMPLEX**)

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- Very **OLD** —> replica should be evolved for a long time, i.e. approximations and numerical errors may propagate and accumulate (need for a way to control the accuracy of the simulation)

However, we can rely on some simplifying (yet reasonable) statements…

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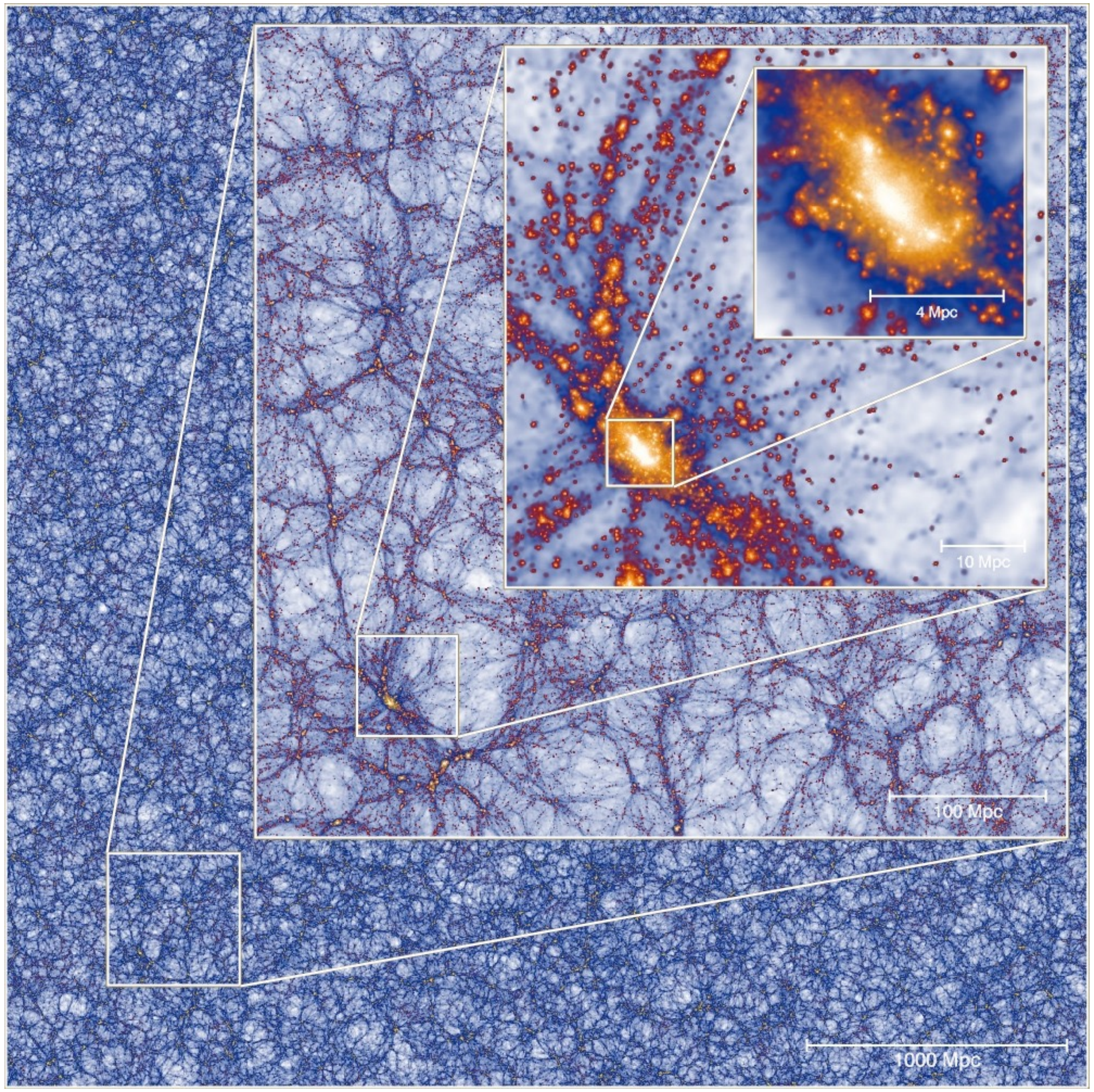

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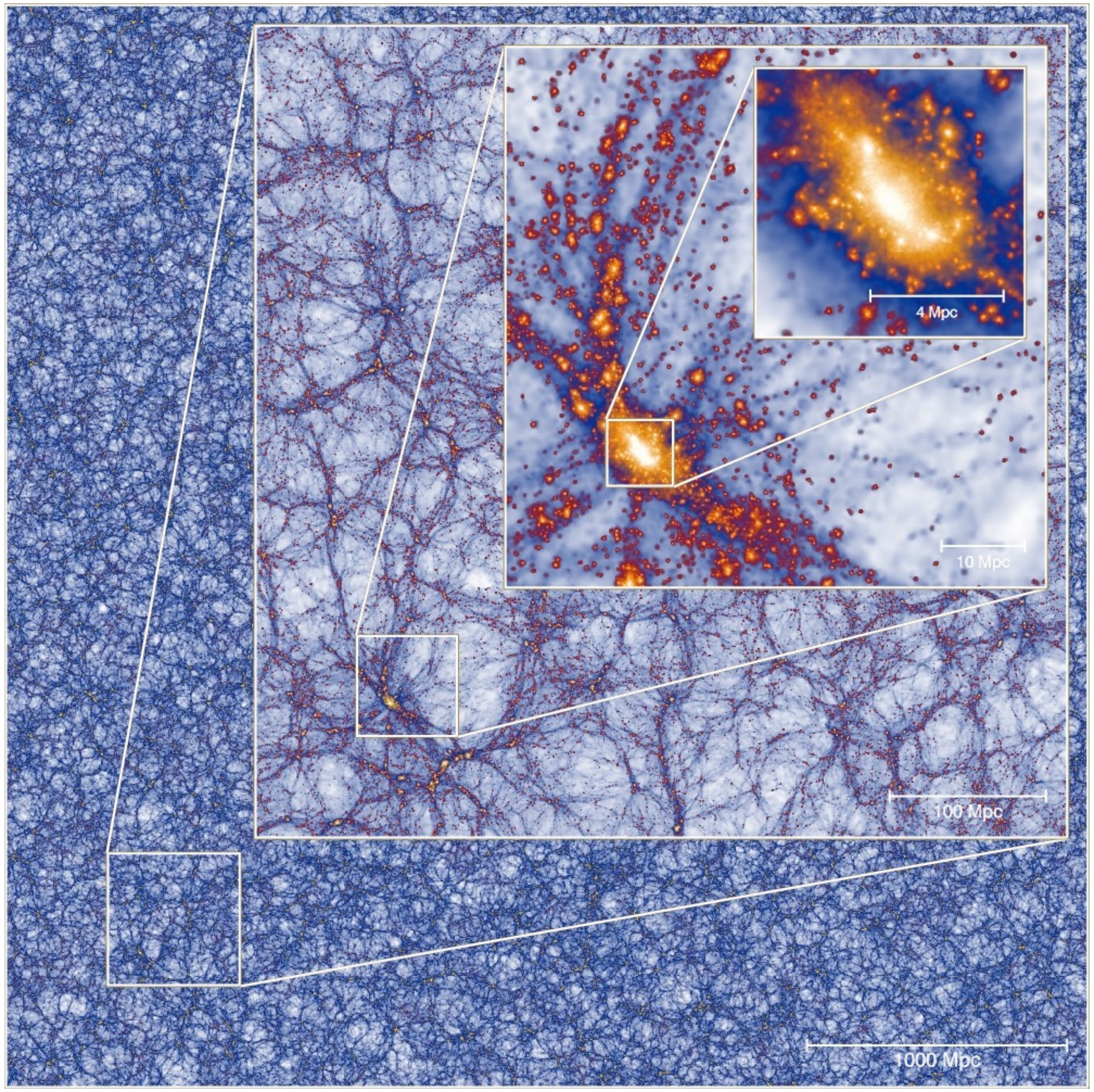

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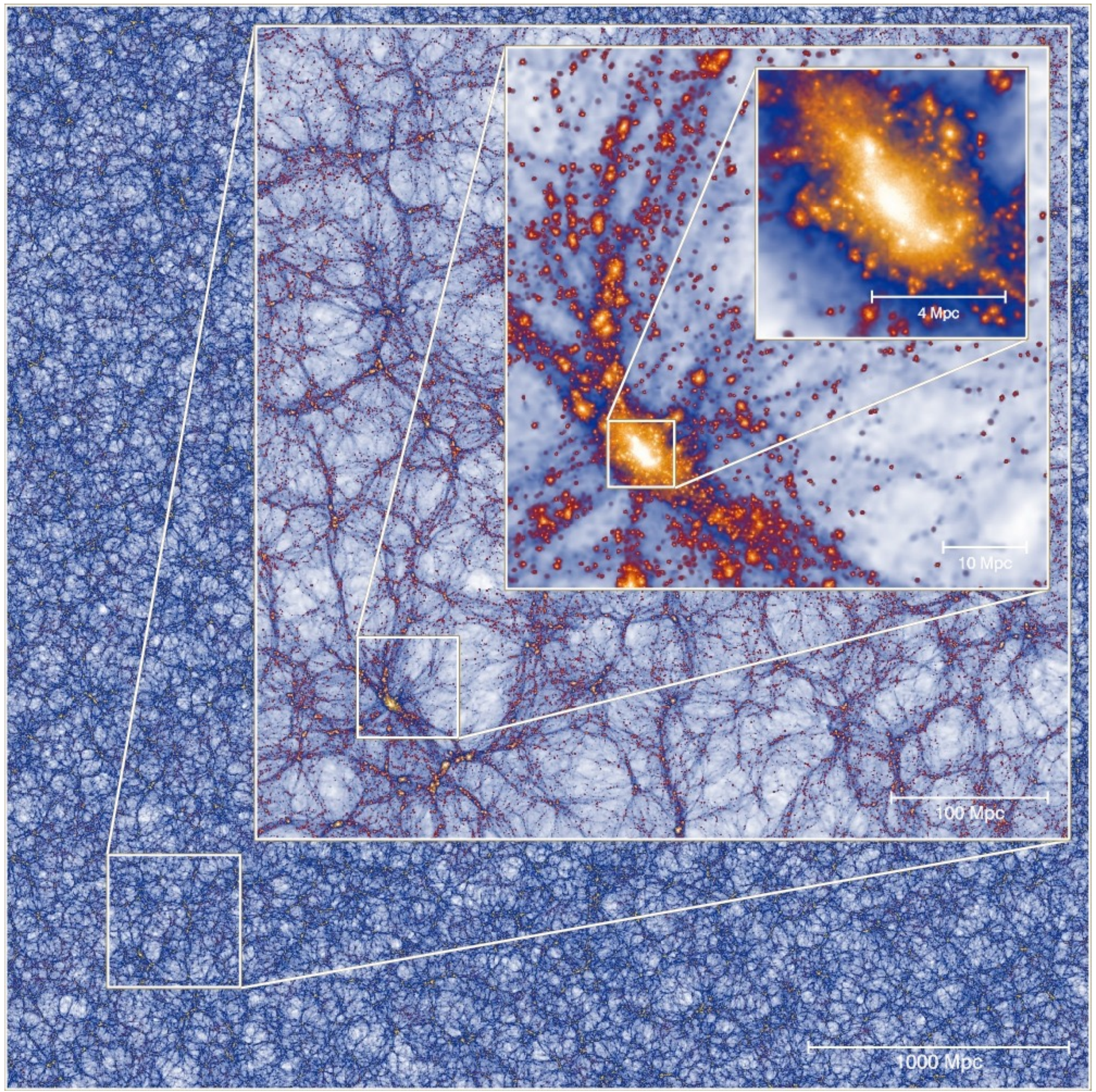

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Bad news!

The homogeneity scale is quite large, a few hundred comoving Mpc

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Bad news!
Gravity is a long-range non-screenable force… we will have to solve a global problem (where every part of the system interacts with any other part)

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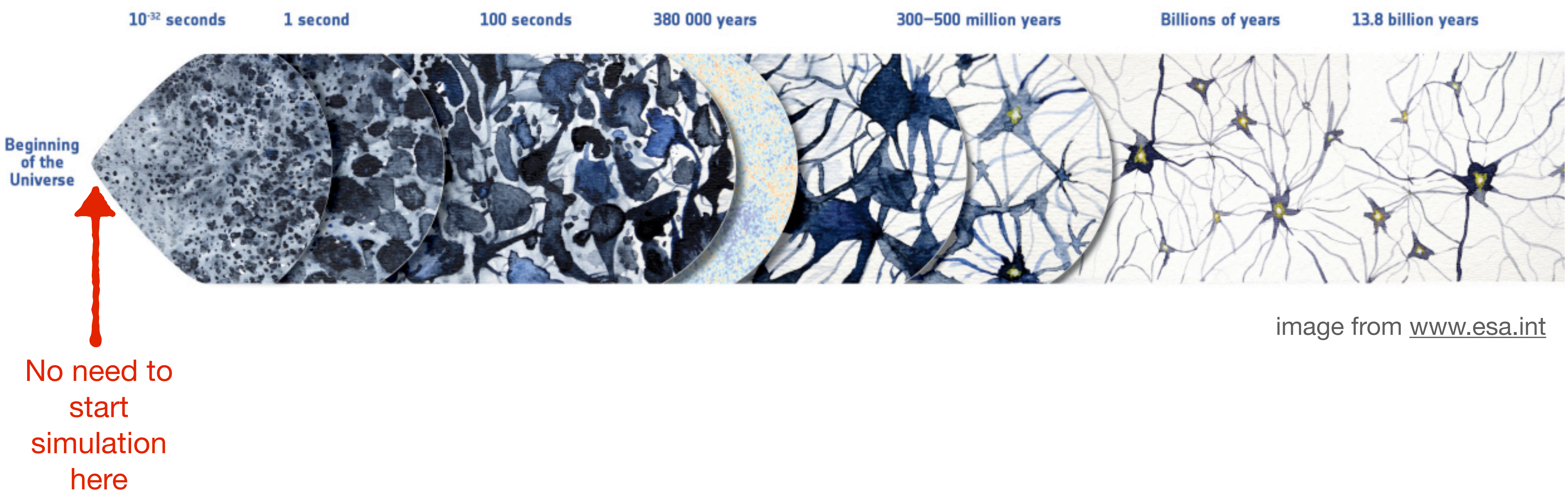

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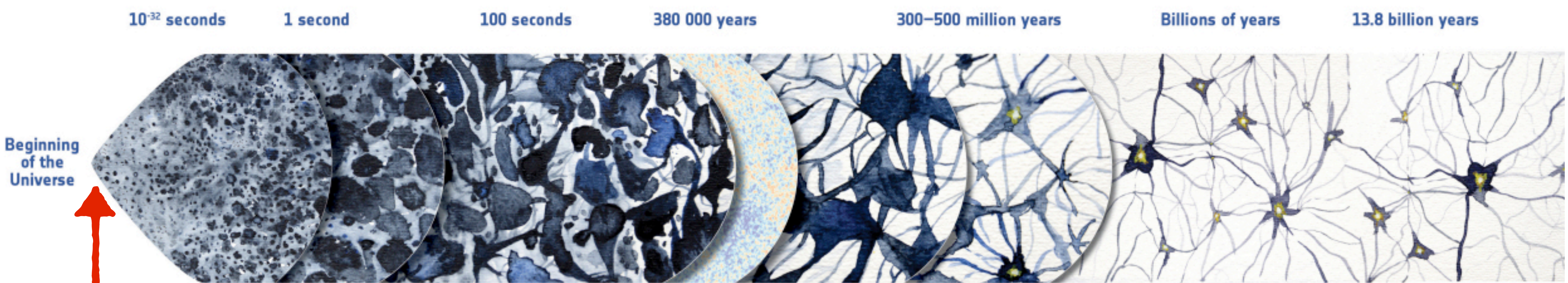

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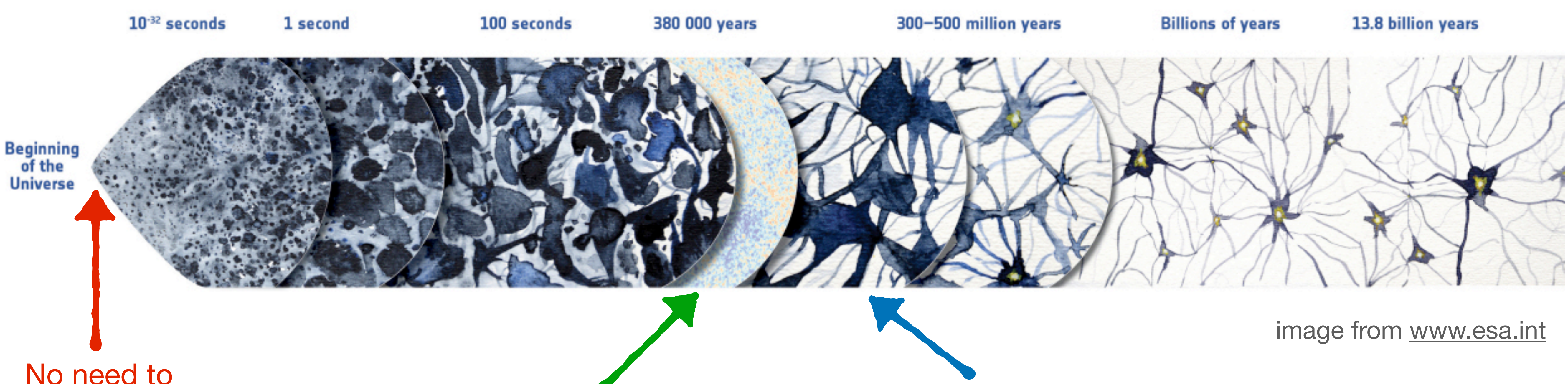

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To start simulation somewhere here

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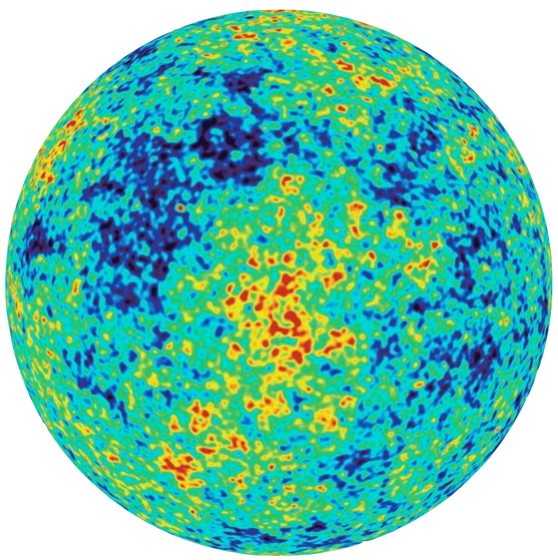
$z_{\rm CMB} \approx 10^3, a_{\rm CMB} \approx 10^{-3}$



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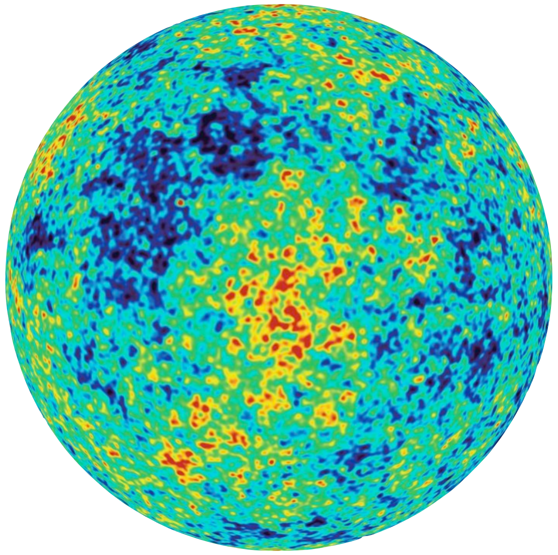

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Structures in the present-day Universe

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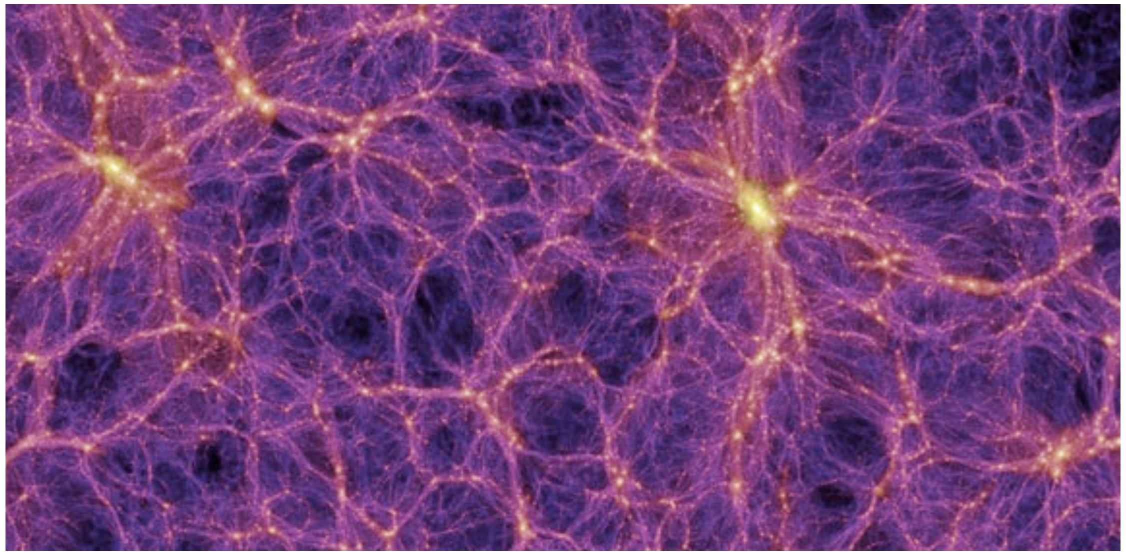

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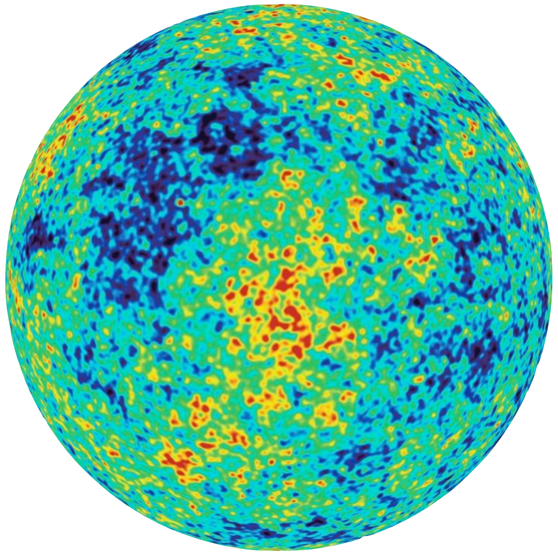

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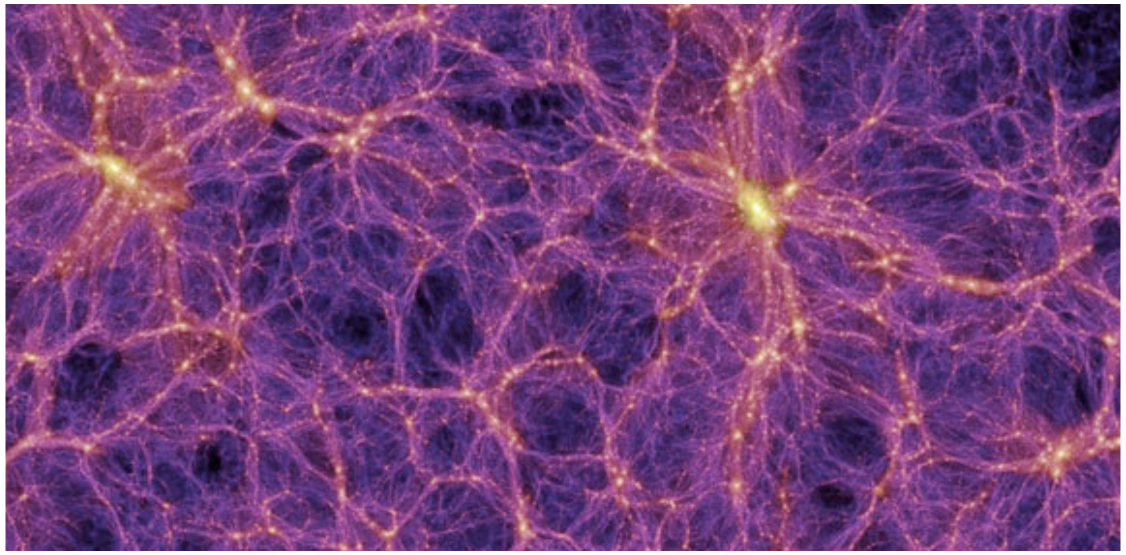

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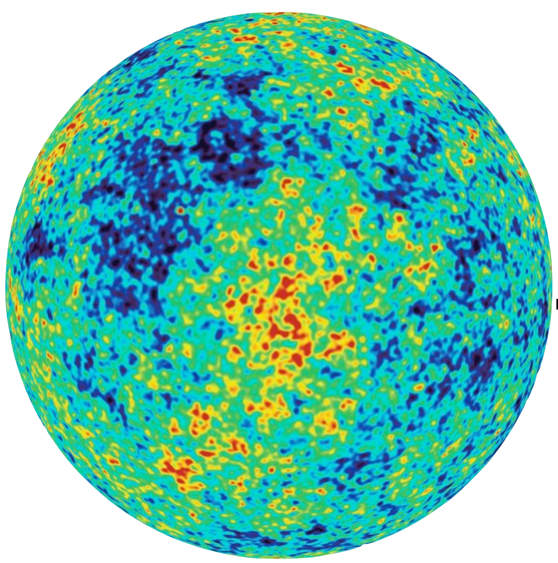

gravity →

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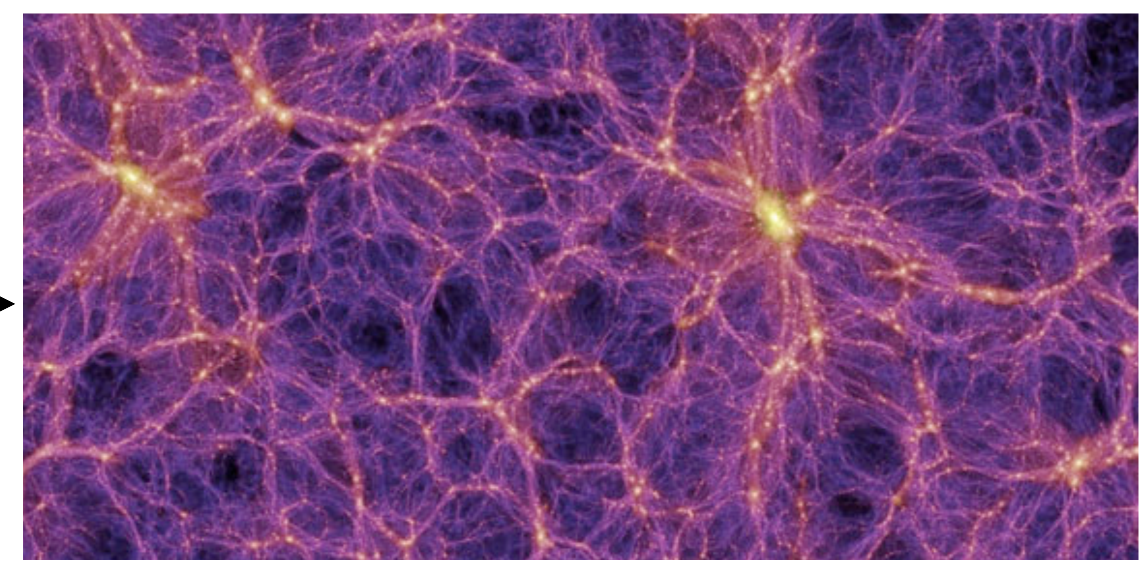

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evolve the density field of the Universe under the effect of gravity from the point where we can predict it analytically to today

How?

Discretisation Approaches

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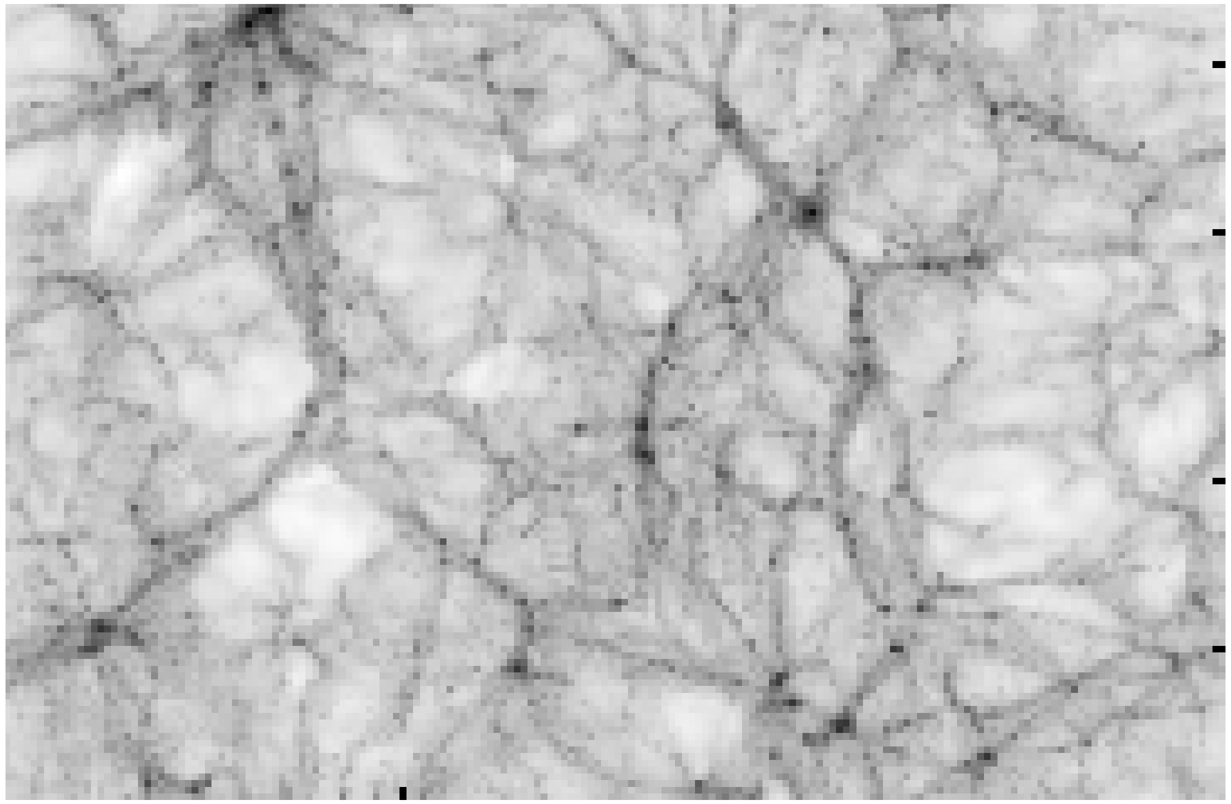

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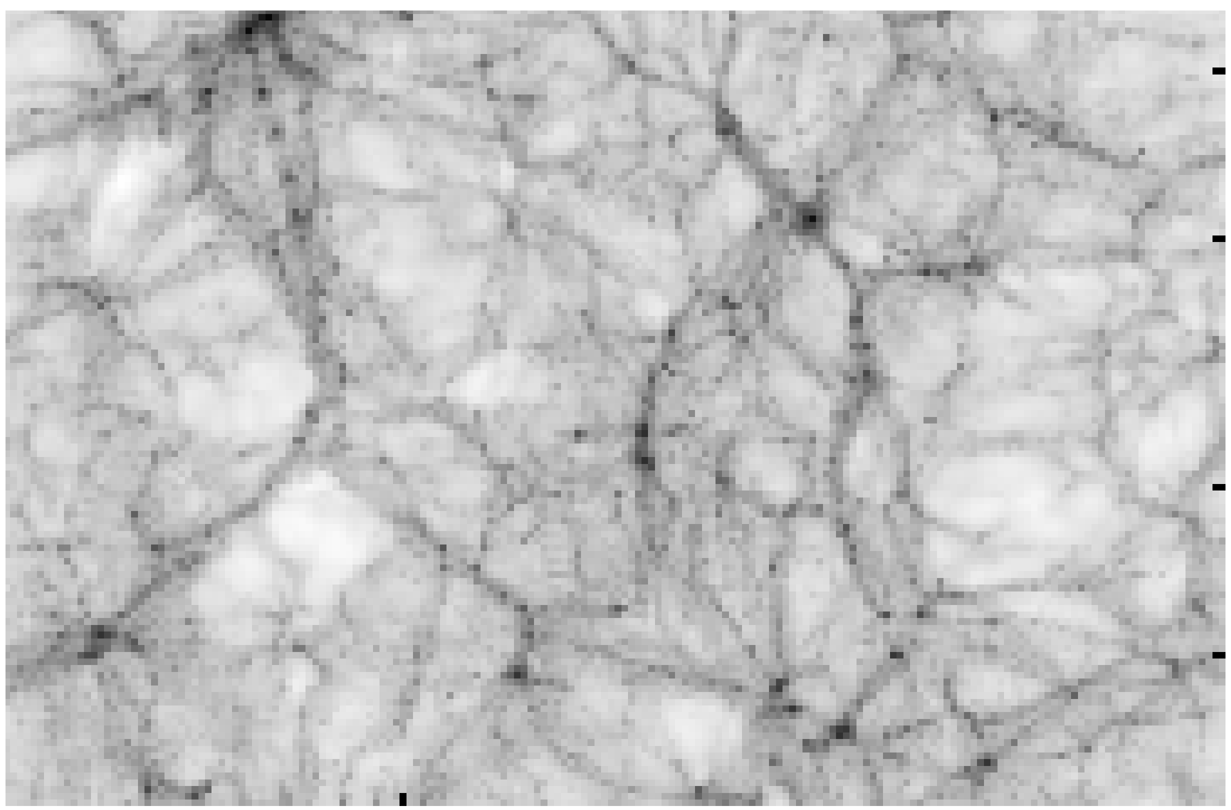
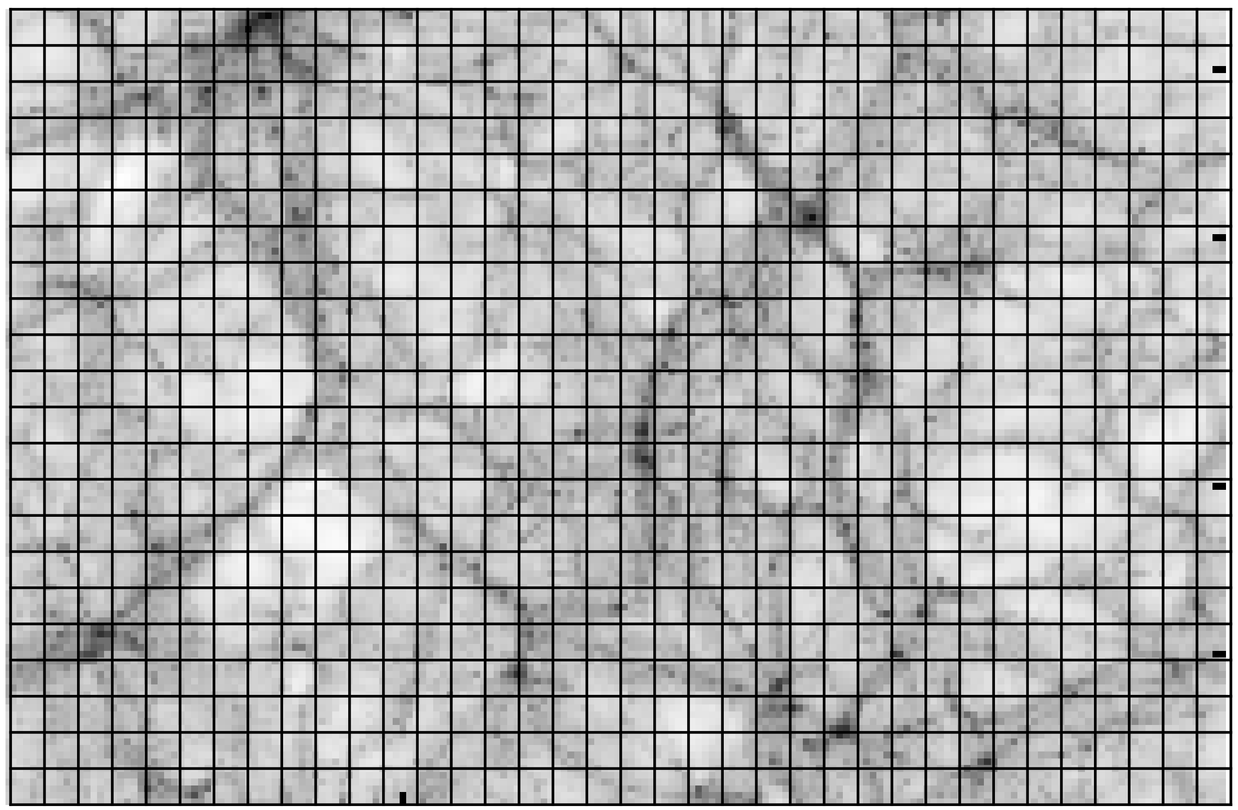
Two possible approaches:

1) Discretise **space** —> Eulerian methods

Density field



Density on a grid (mapping)



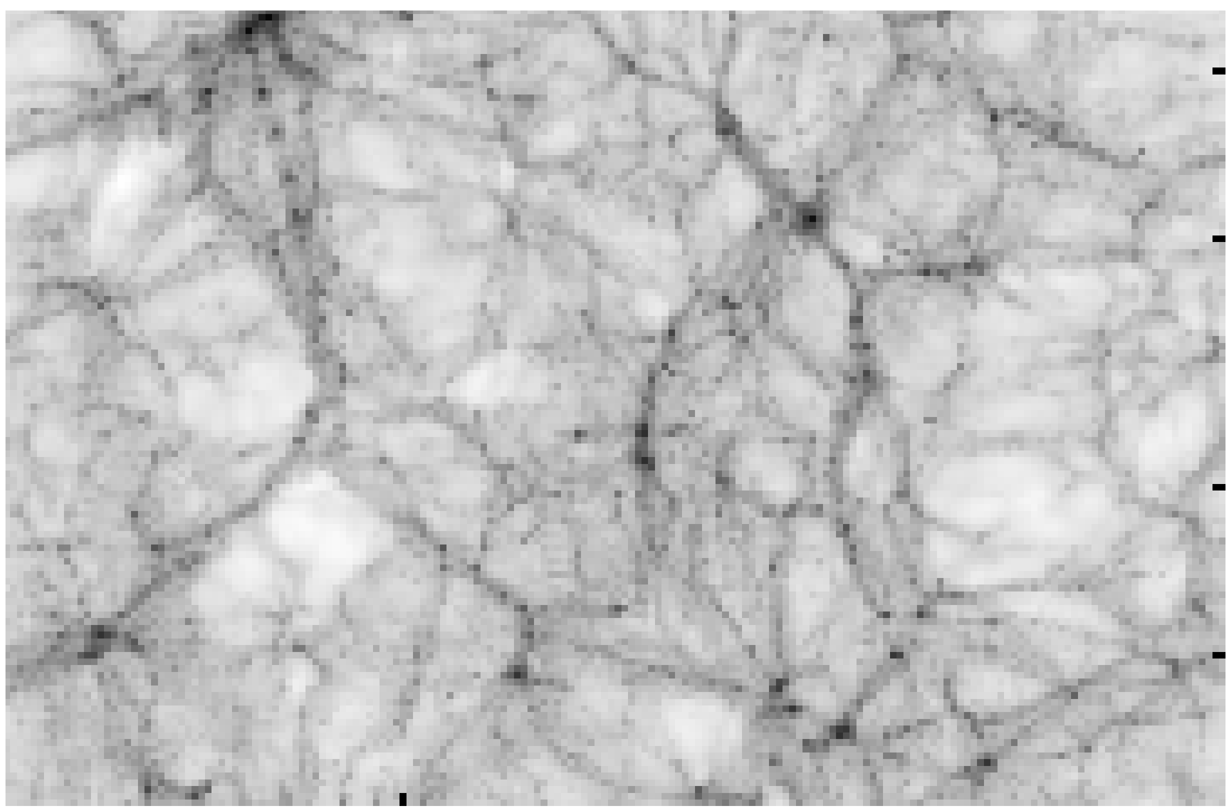
Discretisation approaches

To represent and evolve the 3D density field under its own self-gravity using numerical methods we first need to discretise the system.

Two possible approaches:

1) Discretise **space** —> Eulerian methods

Density field



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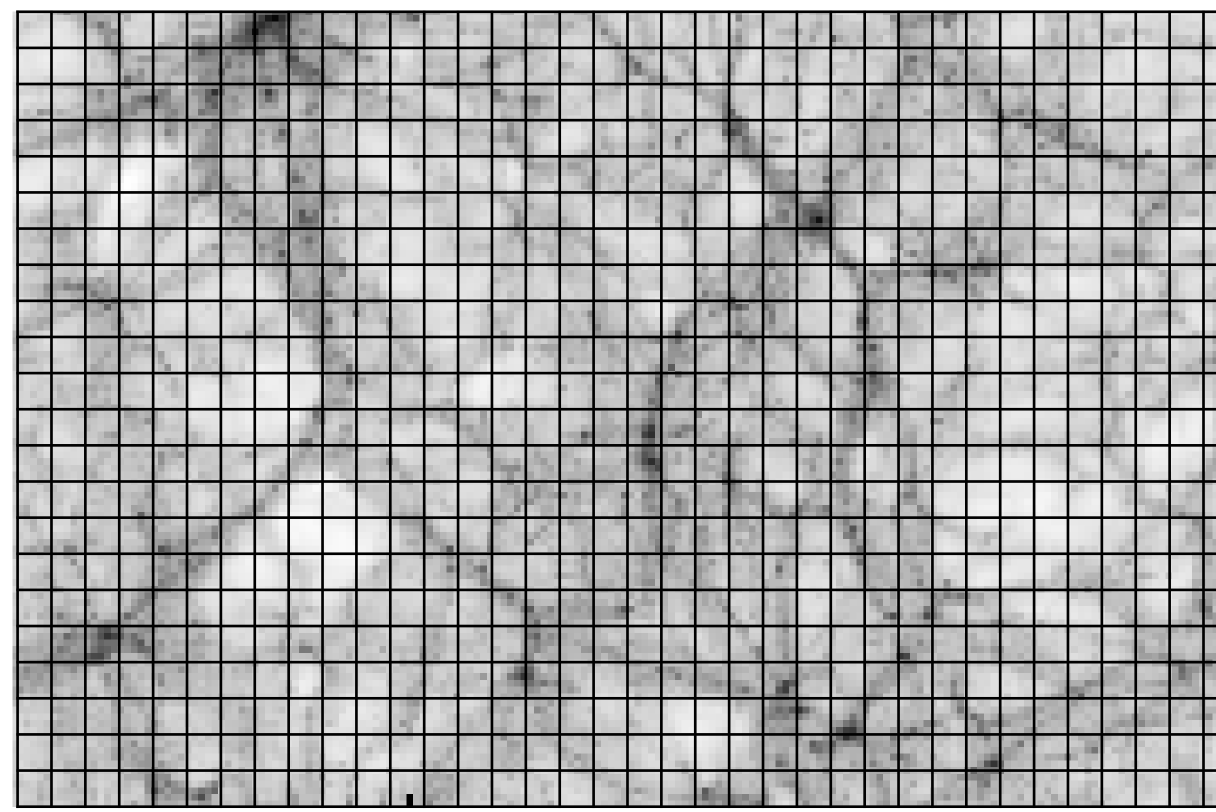

i.e. coarse-graining the field on a given set of space-filling volume elements

Discretisation approaches

To represent and evolve the 3D density field under its own self-gravity using numerical methods we first need to discretise the system.

Two possible approaches:

2) Discretise **mass** —> Lagrangian methods —> N-body approach

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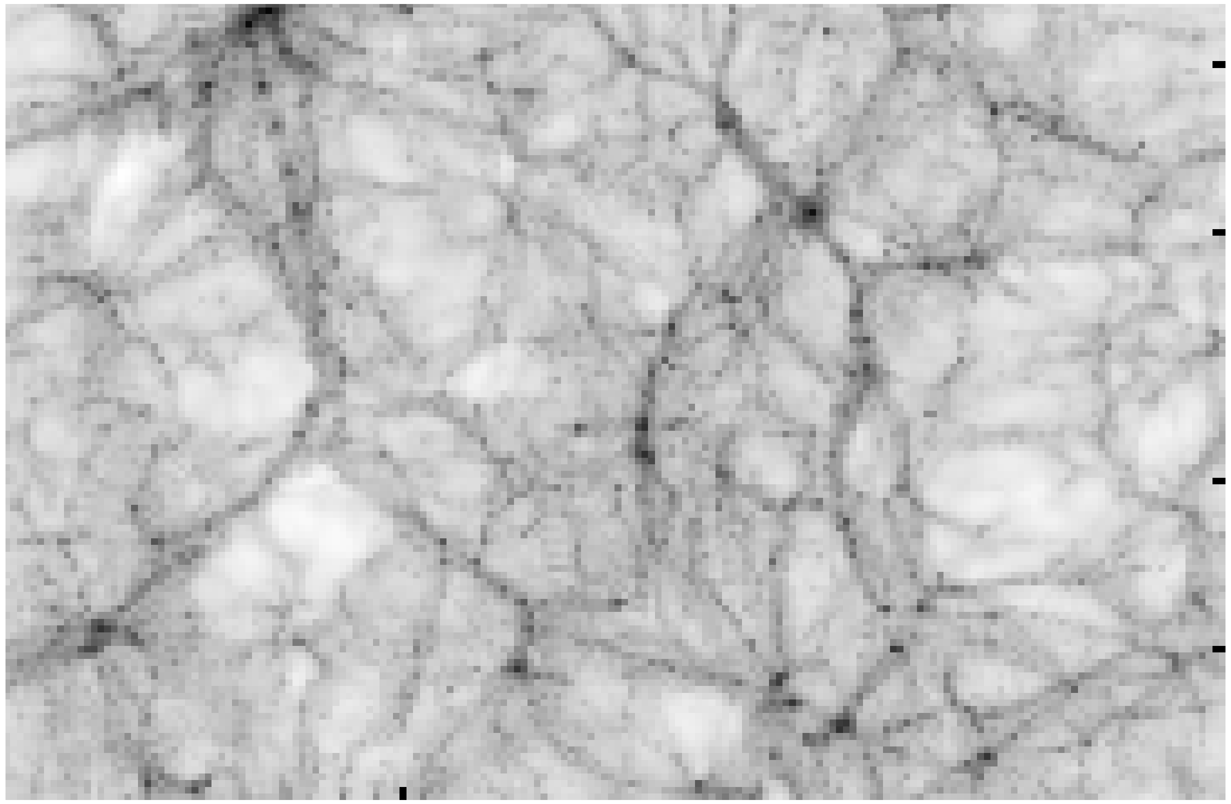

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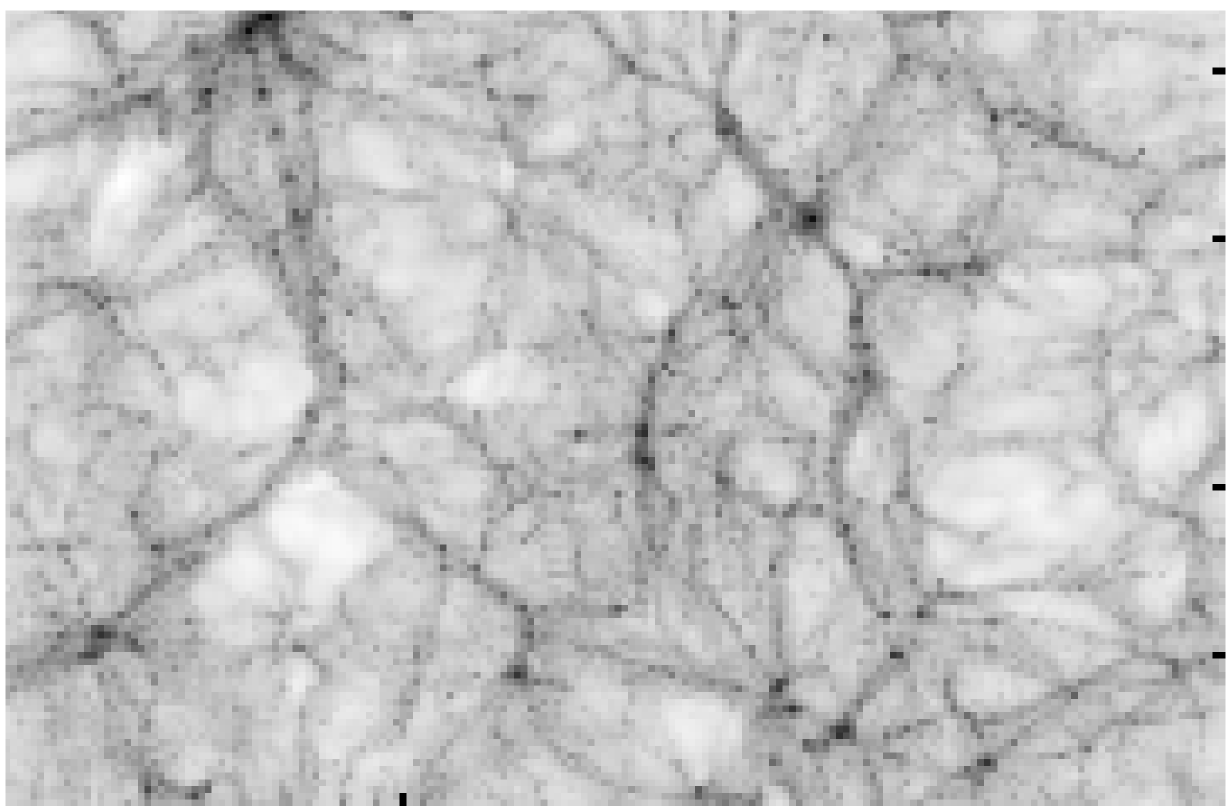

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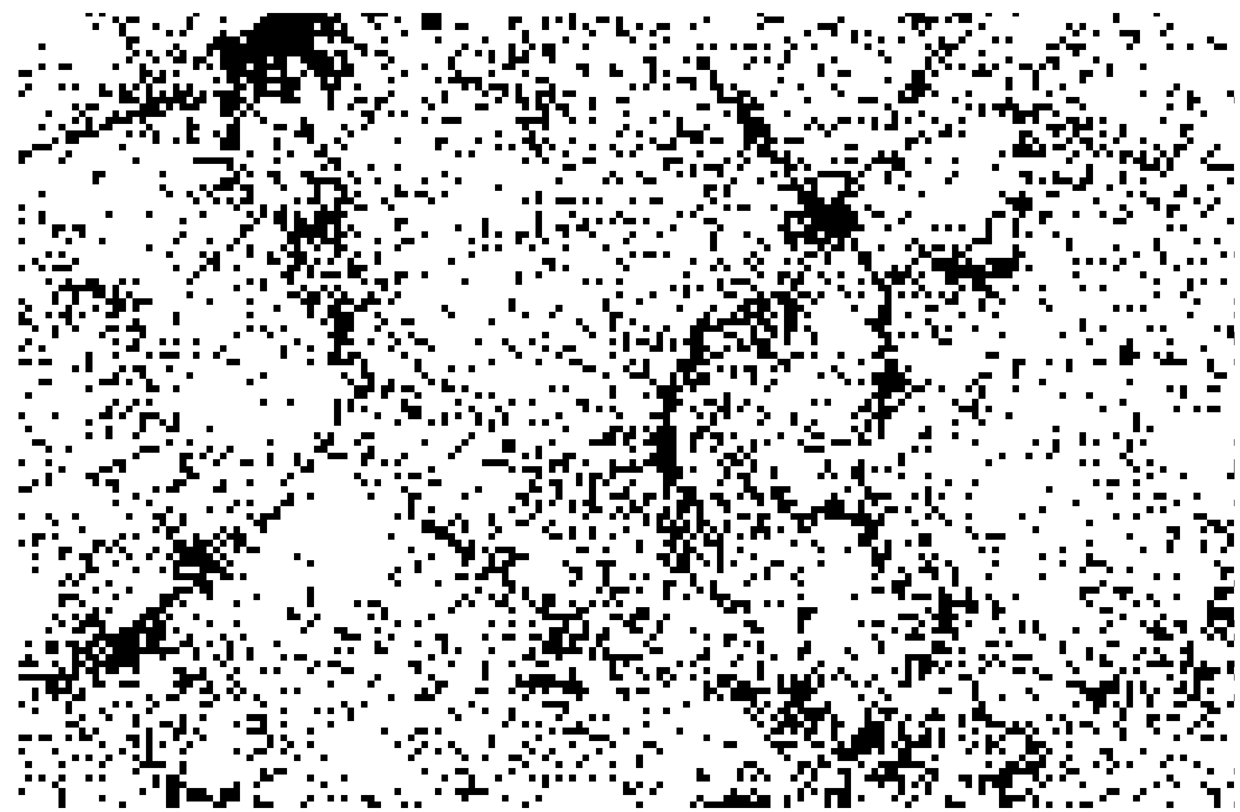

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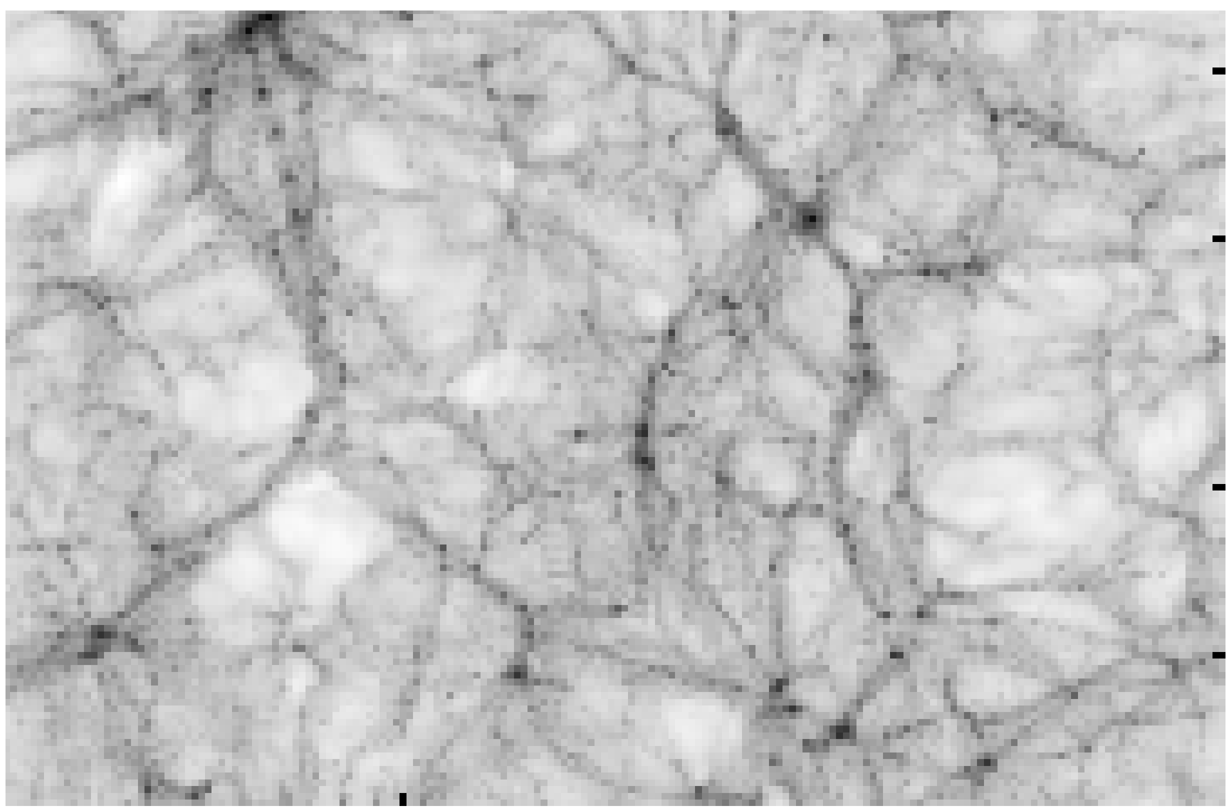

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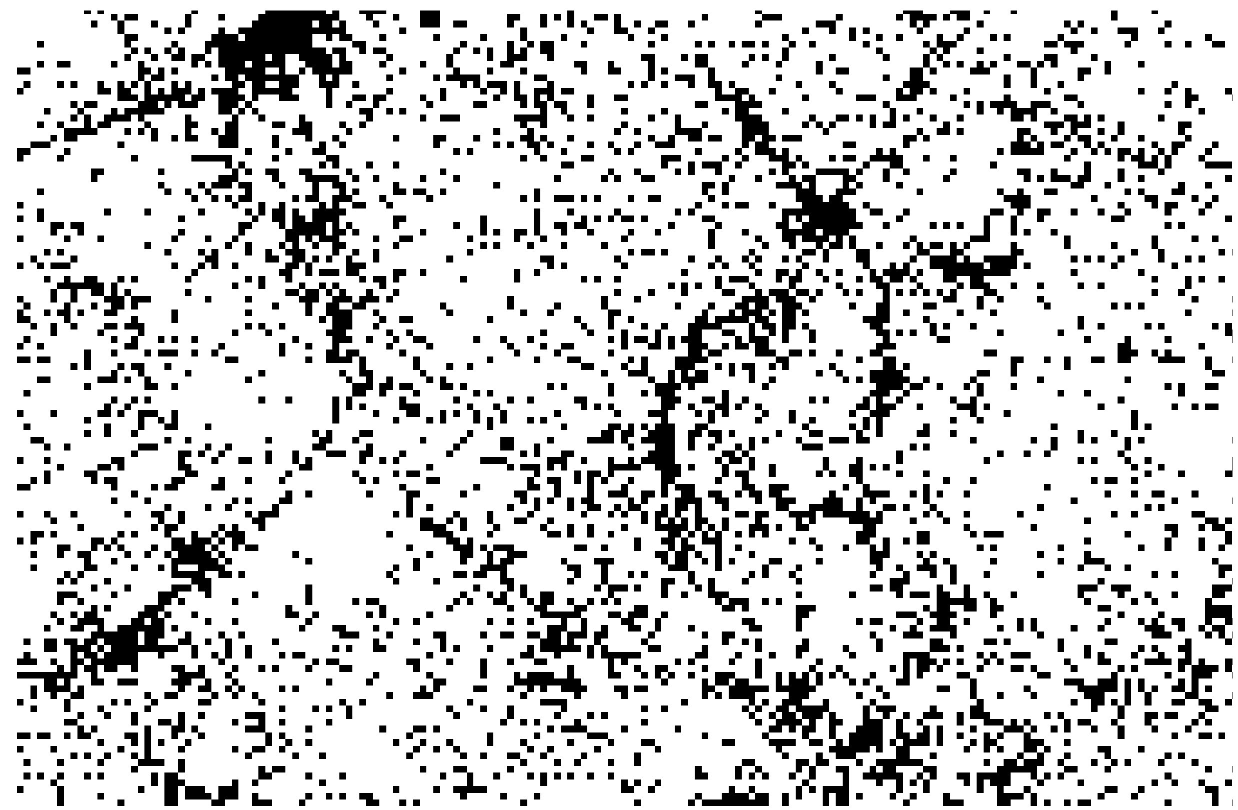

i.e. **sampling the density through a set of discrete point-like tracers** called **"particles"**, which carry a given mass and which have nothing to do (in general) with physical or fundamental particles

How?

Formalising the Problem

The gravitational N-body problem

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where the gravitational potential is given by Poisson's equation:

$$\nabla_r^2 \Phi = 4\pi G \rho$$

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For cosmological systems, one needs to take into account the expansion of space, so that it is convenient to work in comoving coordinates:

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physical coordinate ... comoving coordinate

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This allows to rewrite the equation of motion and the Poisson equation as

$$\dot{\mathbf{v}} = -H(t)\mathbf{v} - \frac{1}{a}\nabla_x\phi$$

$$\nabla_x^2\phi = 4\pi G\bar{\rho}a^2\delta$$

Where ϕ is the peculiar potential and $\delta \equiv \delta\rho/\bar{\rho}$ is the density contrast

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Hubble friction; peculiar potential

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where we now have to write the peculiar potential for a set of N **point-like massive particles**:

$$\phi(\mathbf{x}_i) = -\frac{1}{a}\sum_{j \neq i} \frac{G m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

How?

The 3 main steps

The steps of an N-body simulation

1. **Initial Conditions:**

   The gravitational N-body problem amounts to integrate a system of first order differential equations:

$$\dot{\mathbf{x}}_i = \mathbf{p}_i / a^2$$

$$\dot{\mathbf{p}}_i = -\nabla_x \phi(\mathbf{x}_i)$$

   We therefore need initial conditions for $\mathbf{x}$ and $\dot{\mathbf{x}}$ for all the particles in the system

The steps of an N-body simulation

2. **Gravitational Solver**

It will be the core of our calculation, and requires to compute the source term for the acceleration equation for any given configuration of the particles' distribution.
In other words, it is the method to obtain the source term on the RHS of the differential equations:

$$\dot{\mathbf{p}}_i = -\nabla_x \phi(\mathbf{x}_i)$$

which requires computing the peculiar gravitational potential

$$\phi(\mathbf{x}_i) = -\frac{1}{a}\sum_{j\neq i}\frac{Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

at every particle's position

The steps of an N-body simulation

3. **Time integration**

   Once the source term (i.e. the acceleration on all particles) has been computed, and the initial configuration of the system is known (particles' positions and velocities at a given time) the dynamics must be integrated by **moving the system forward in time**, thereby updating positions and velocities of all particles:

$$\mathbf{x}(t) \rightarrow \mathbf{x}(t + \Delta t)$$

$$\mathbf{p}(t) \rightarrow \mathbf{p}(t + \Delta t)$$

How?

Setting up initial conditions

Step 1: initial conditions

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Which is solved by the growth function $D(a) \, : \delta(a) = D(a)\delta_0$

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For $\delta \sim 1$ this treatment breaks down ($\delta > 1$ would imply negative densities), but as long as $\delta \ll 1$ it provides a way to obtain δ from observations (of $P(k)$). How?

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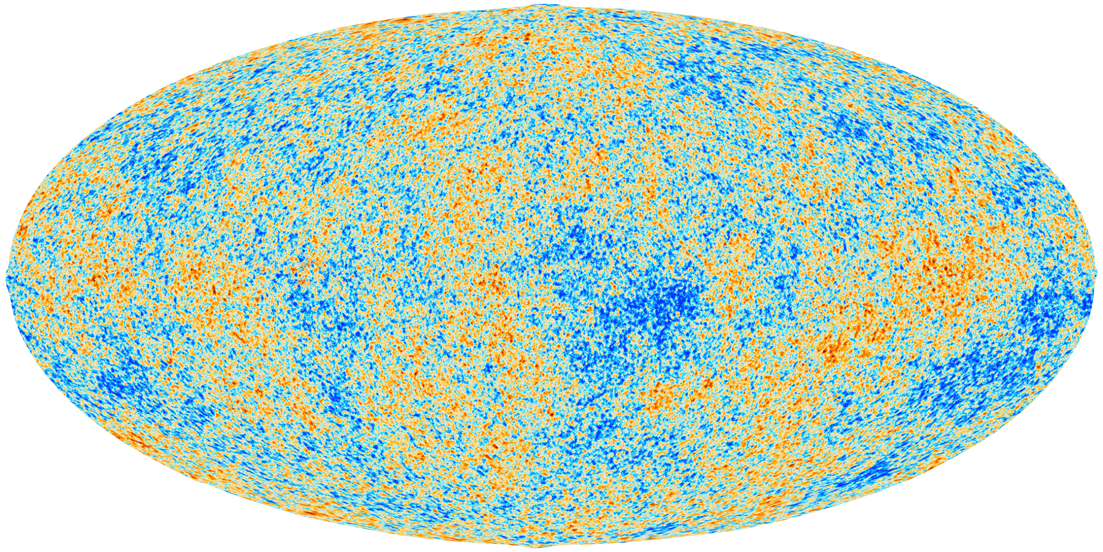

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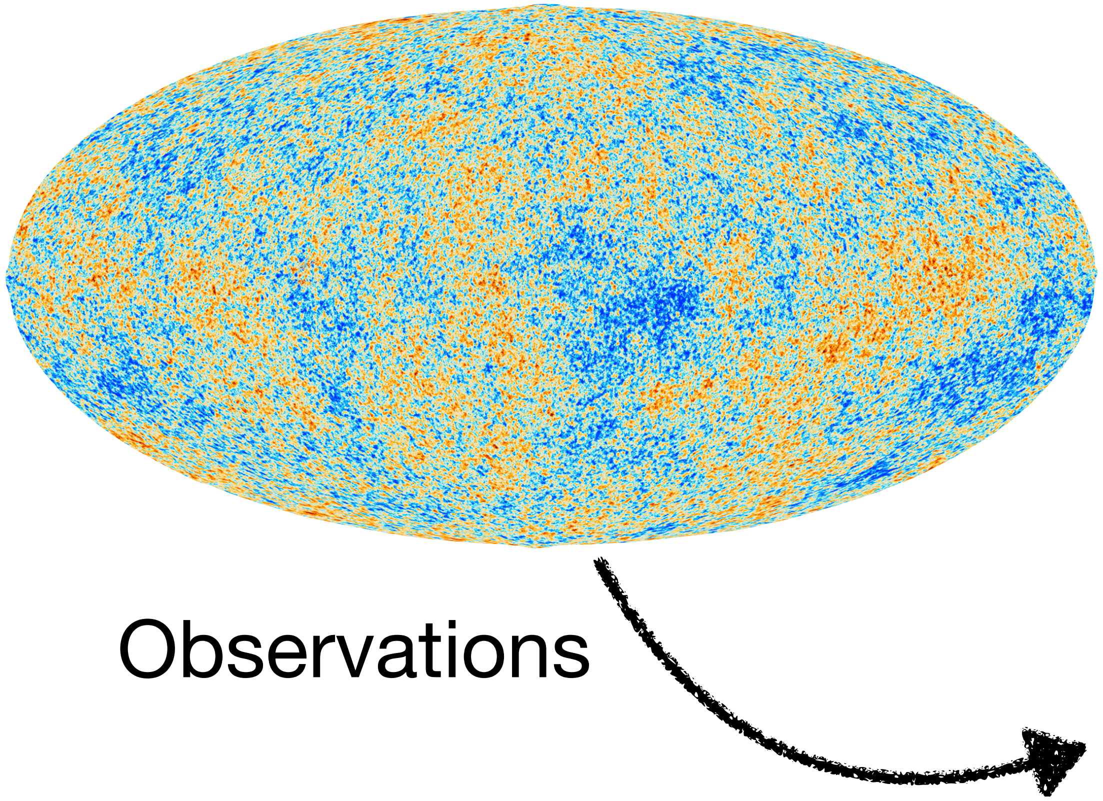
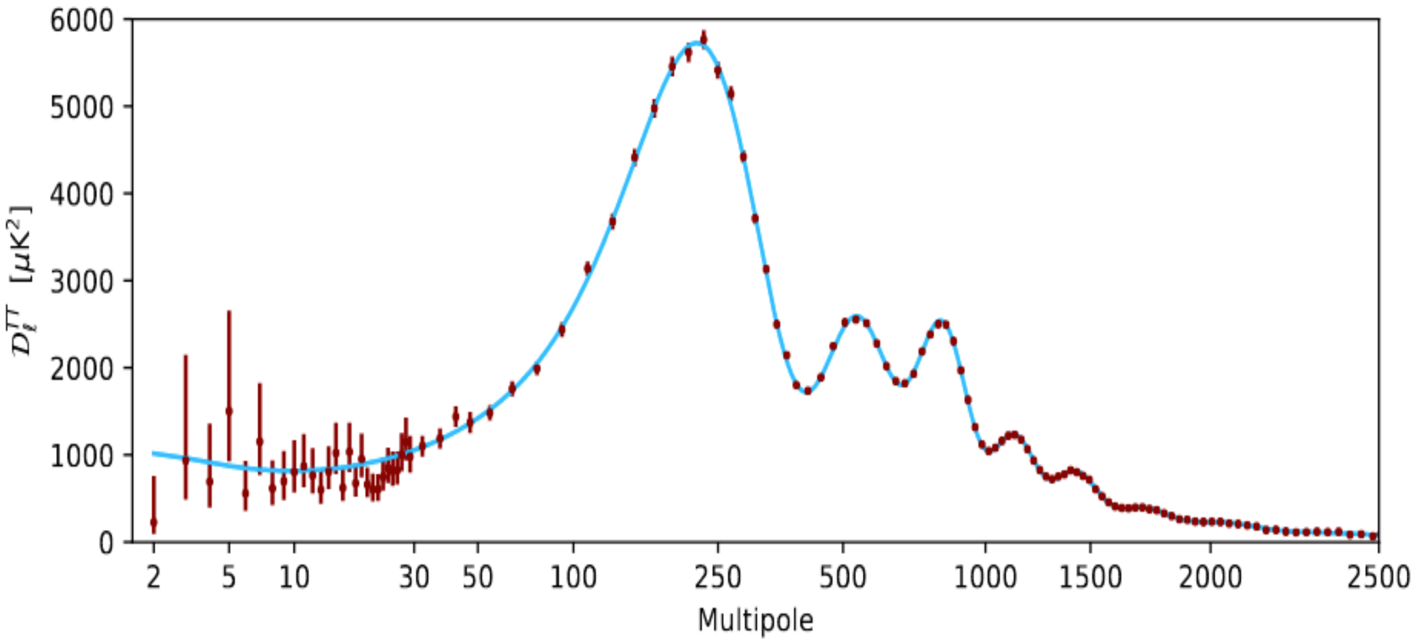


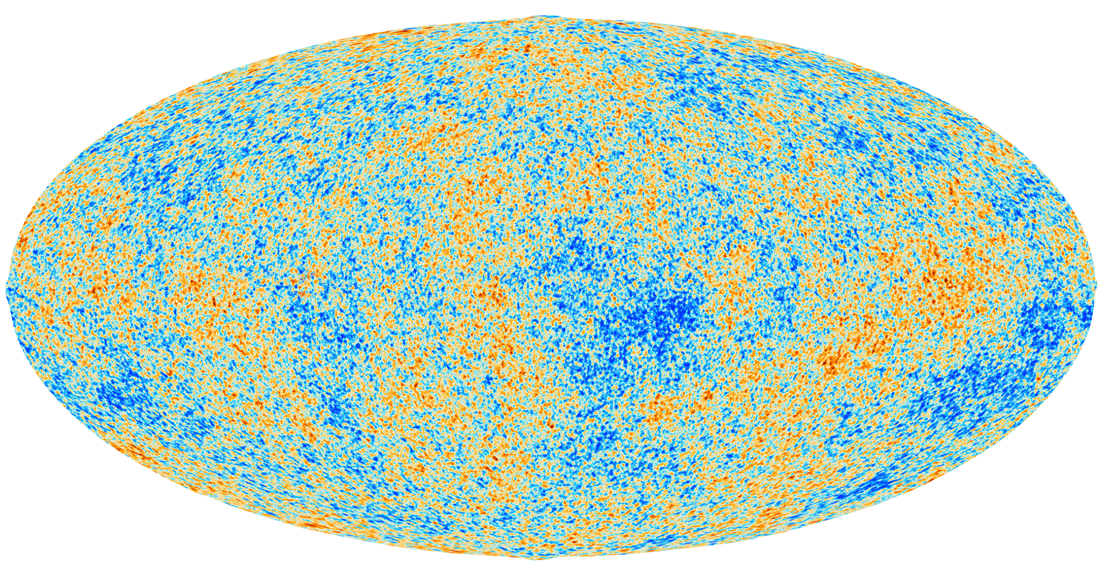
Observations



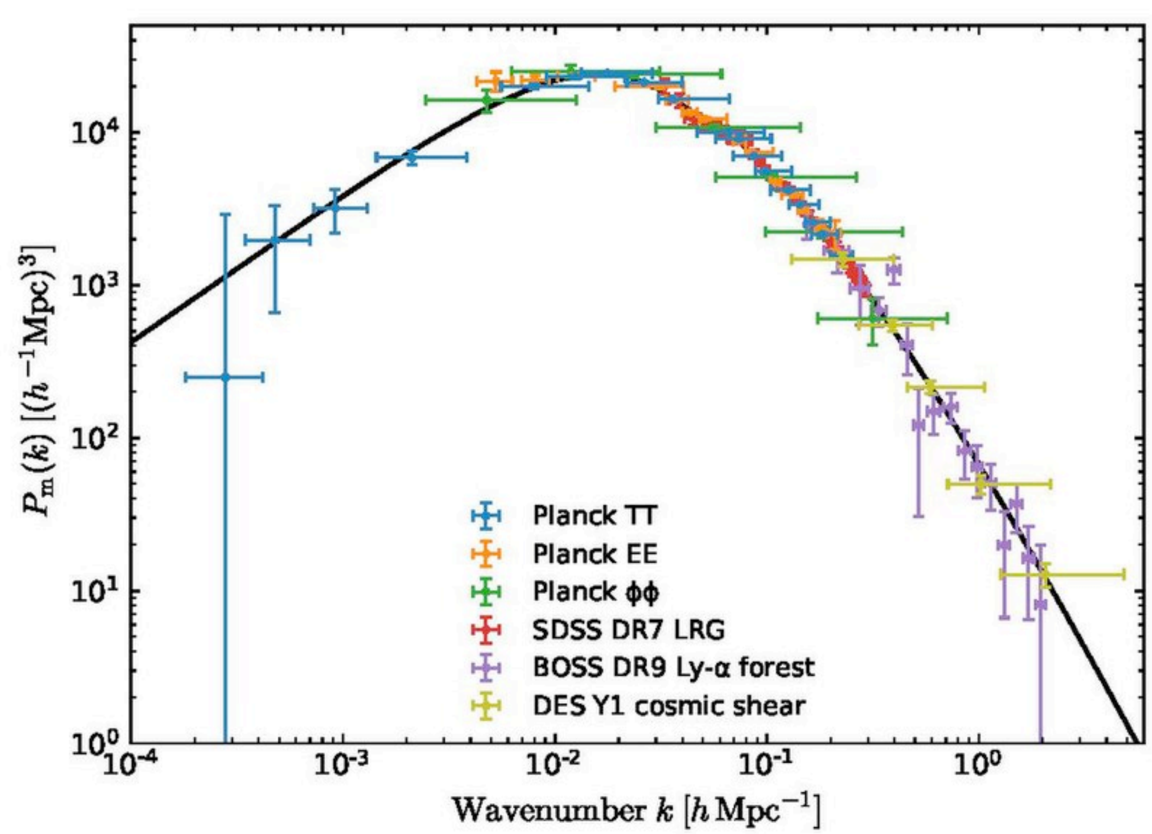
Plots: Planck Collaboration

Step 1: initial conditions

We want to translate the temperature fluctuations observed in the CMB (well in the linear regime, $\delta \ll 1$) into initial conditions for particles' positions and velocities. First, convert the angular power spectrum of temperature fluctuations in a 3D density power spectrum



Observations
+
Modelling



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This implies

$$\nabla^2\phi = \frac{D(a)}{a}\nabla^2\phi_0 \Rightarrow \phi(a) = \frac{D(a)}{a}\phi_0$$

So that in matter domination ($D(a) \sim a$) one has $\phi(a) \sim \mathrm{const.}$

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and integrate it:

$$a\mathbf{v} = -\int \nabla\phi dt = -\nabla\phi_0 \int \frac{D(a)}{a} dt$$

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so to get:

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Take the perturbations evolution equation

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and substitute its solution $D(a)$:

$$\ddot{D} + 2H\dot{D} = 4\pi G\frac{\bar{\rho}_0}{a^3}D \quad \rightarrow \quad \frac{1}{a^2}\frac{\partial(a^2\dot{D})}{\partial t} = 4\pi G\frac{\bar{\rho}_0}{a^3}D$$

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Combine the results of B and C

$$\mathbf{v} = -\frac{\nabla\phi_0}{a}\int\frac{D(a)}{a}dt \qquad\qquad \frac{a^2\dot{D}}{4\pi G\bar{\rho}_0} = \int\frac{D(a)}{a}dt$$

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To get:

$$\dot{\mathbf{x}} = \frac{\mathbf{v}}{a} = -\frac{\nabla\phi_0}{a^2}\int\frac{D(a)}{a}dt = -\frac{\nabla\phi_0}{a^2}\frac{a^2\dot{D}}{4\pi G\bar{\rho}_0} = -\frac{\dot{D}\nabla\phi_0}{4\pi G\bar{\rho}_0}$$

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E

Step 1: initial conditions

Once we have translated the CMB temperature fluctuations into a matter density power spectrum, we need to impose this spectrum on a set of N particles. This can be done through Zel'dovich approximation.

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Now integrate the latter to get the Zel'dovich approximation:

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so that we can define the displacement field as

$$\psi \equiv \mathbf{x} - \mathbf{x}_0 = -\frac{a\nabla\phi(a)}{4\pi G\bar{\rho}_0}$$

that allows to evolve particles' positions in the linear regime

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then substitute:

$$\psi_k = \frac{a(i\mathbf{k})}{4\pi G\bar{\rho}_0}\frac{4\pi G\bar{\rho}_0}{ak^2}\delta_k = i\mathbf{k}\frac{\delta_k}{k^2}$$

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where $A_k e^{i\phi_k} = B_1 + iB_2$ and $B_1 \, , B_2$ are drawn from a Gaussian distribution. Finally, one gets:

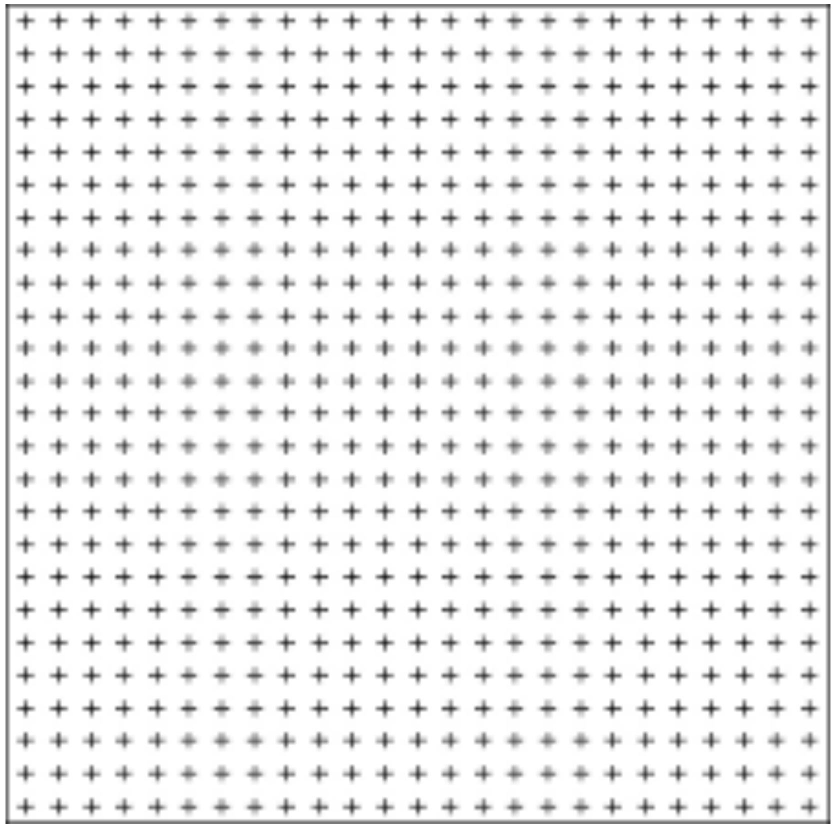
$$\mathbf{x} - \mathbf{x}_0 = \psi \rightarrow (\mathbf{x} - \mathbf{x}_0)_k = i\mathbf{k}\frac{\delta_k}{k^2}$$

$$\dot{\mathbf{x}} = \frac{\dot{D}}{D}\psi \rightarrow \mathbf{v}_k = a\frac{\dot{D}}{D}\psi_k$$

Step 1: initial conditions

Fourier transforming back to position space we obtain the desired initial conditions for the simulation.

Unperturbed positions
(homogeneous grid)



Initial Conditions
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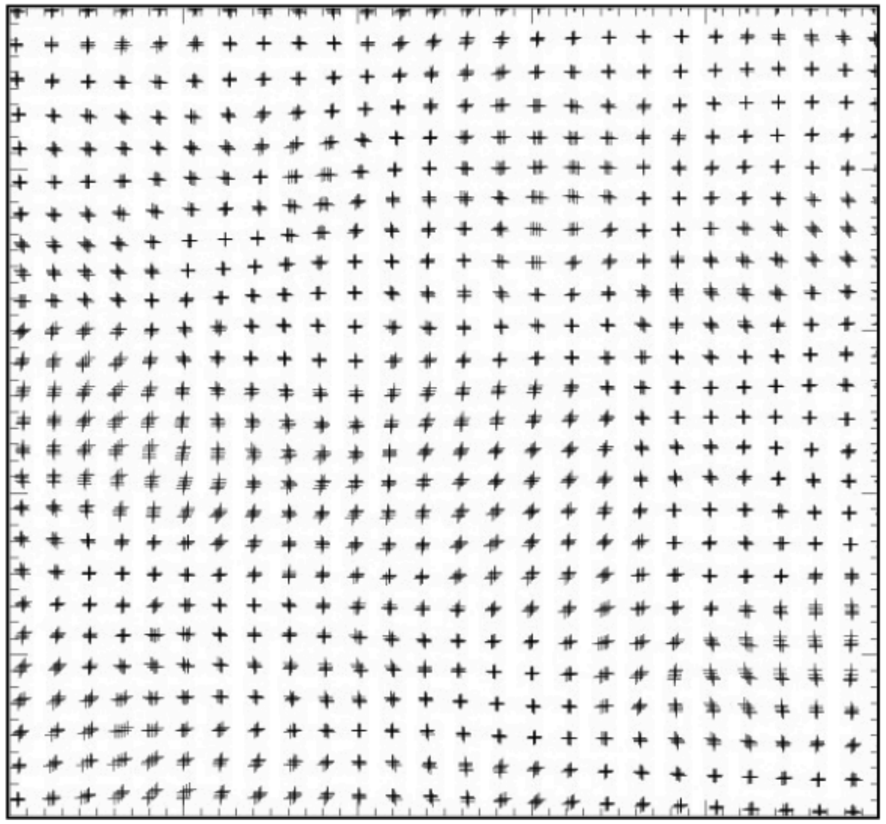

How?

Solving gravity

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Now that we have an initial particle distribution with well-defined values of $\mathbf{x}$ and $\mathbf{v}$ we need to solve the dynamics:

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The system will behave as collision less when evolving over a time span $t \ll t_{\rm relax}$. For dark matter particles in a galaxy one has

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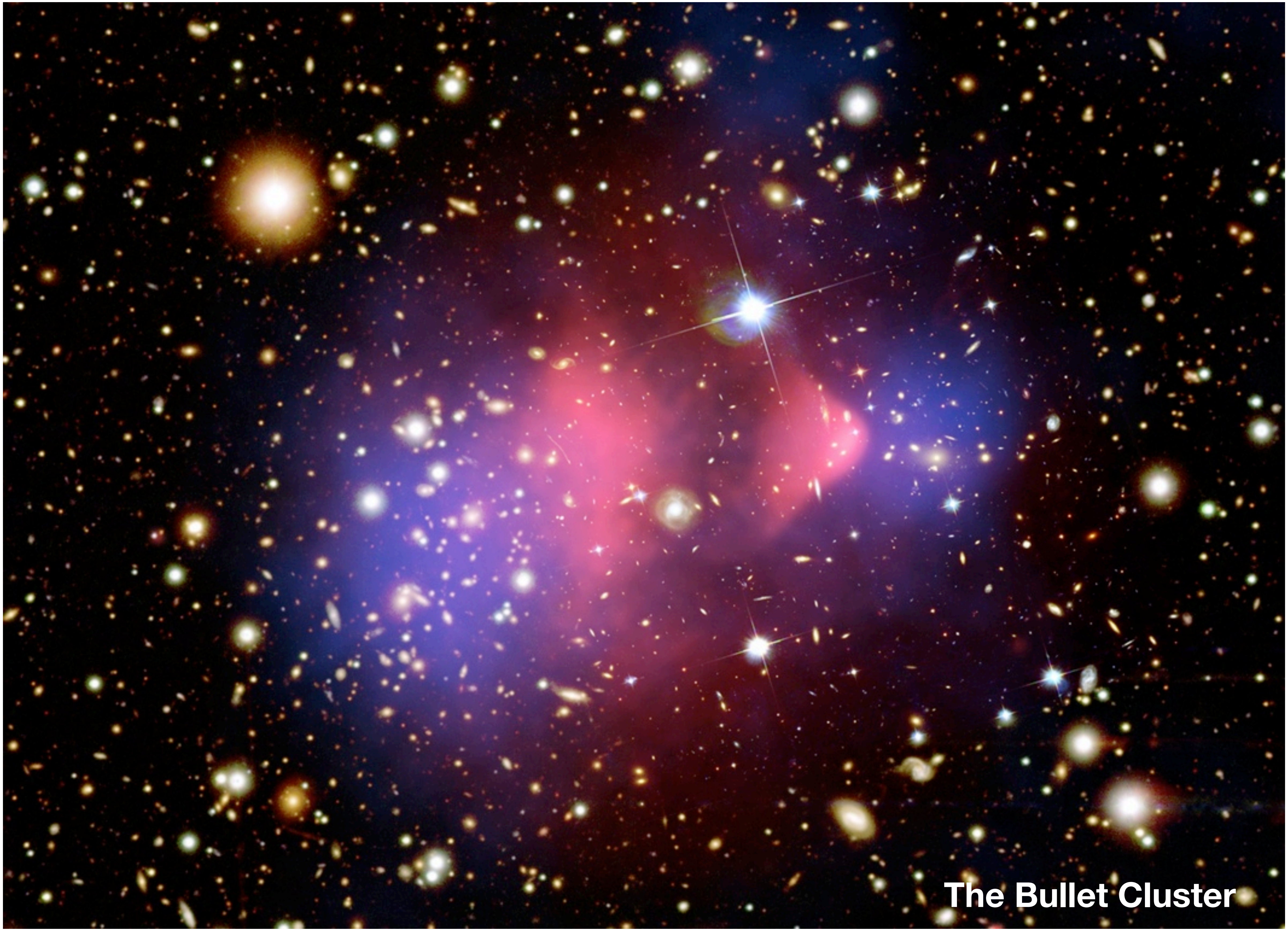
$$N \sim 10^{67} \qquad t_{\mathrm{cross}} \sim 0.1/H_0 \;\Rightarrow t_{\mathrm{relax}} \sim 10^{63}/H_0$$

So dark matter in a galaxy is evolving collisionlessly throughout the whole cosmic history. This is true also for (standard cold) dark matter in any larger cosmic volume

Solving gravity: a note on collisionless systems



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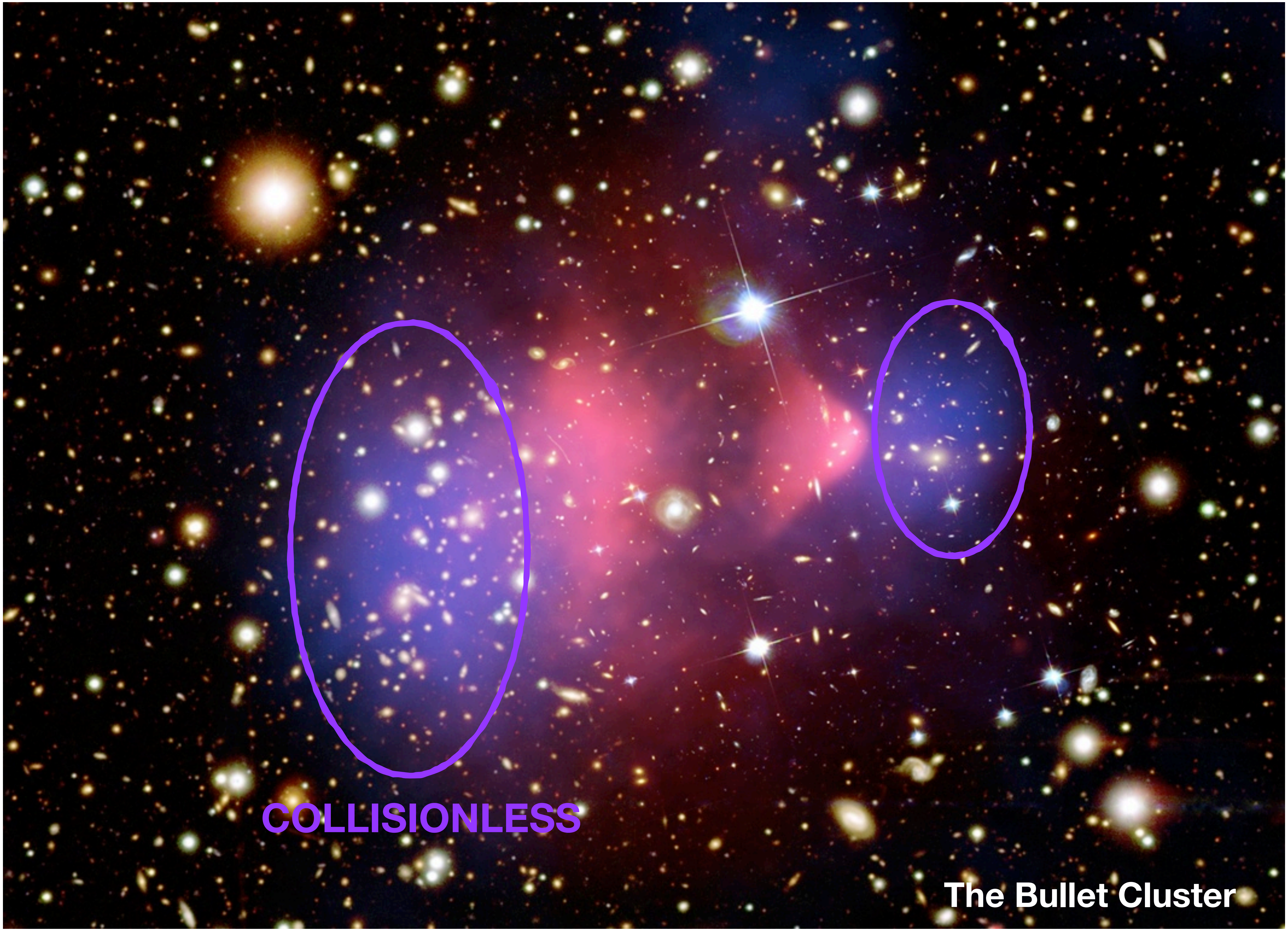

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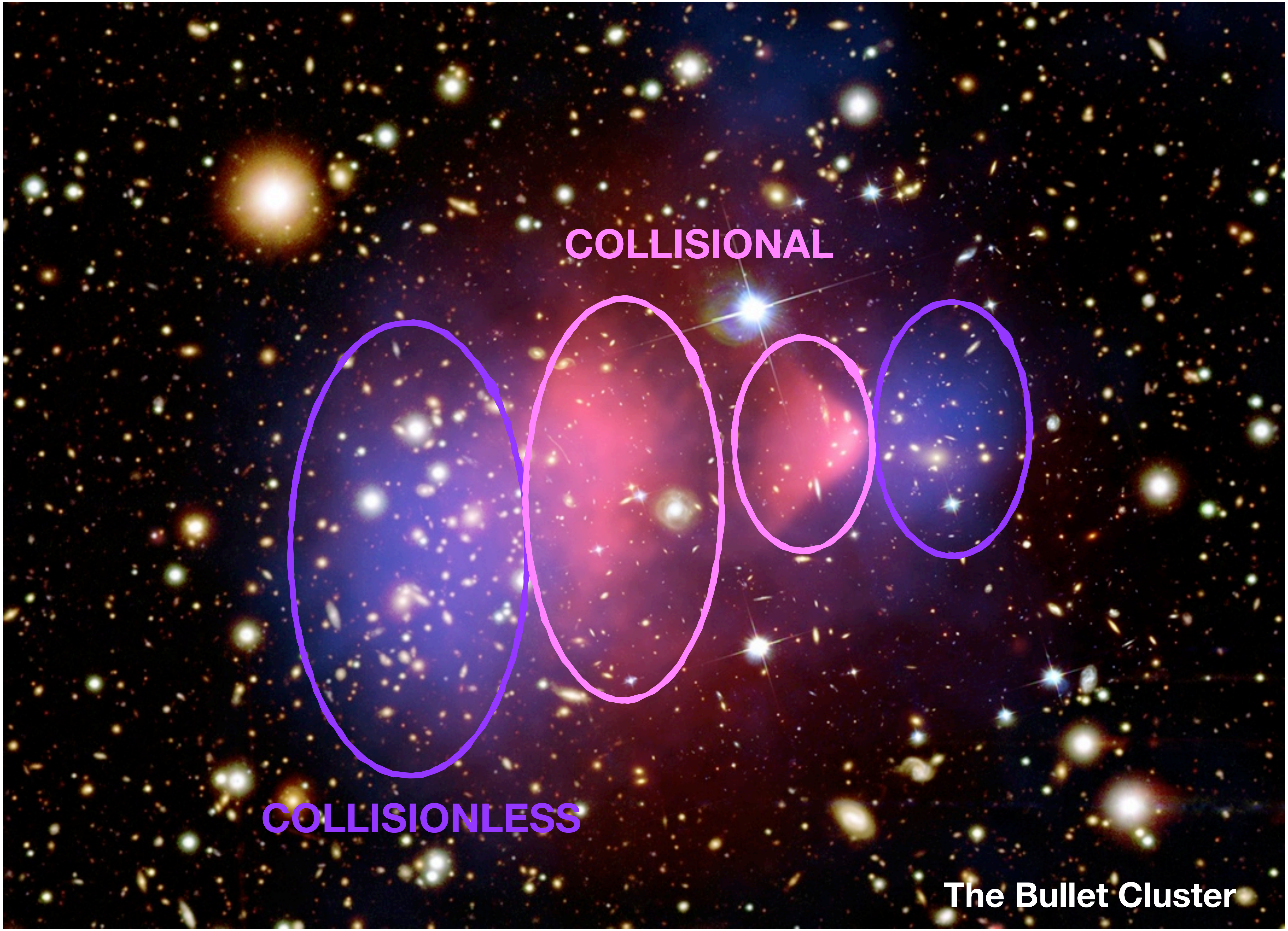

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So we know that (standard cold) dark matter in the Universe is expected to behave as a collision less system. However, we will sample it with mass elements representing a (huge) ensemble of fundamental particles (so N is much lower than in reality):

$$m_\chi \approx 100\,\mathrm{Gev}/c^2$$
$$m_{\rm sim} \approx 10^8 - 10^{10}\,\mathrm{M}_\odot/h$$

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Still, as long as we can ensure $t_{\mathrm{relax}} \gg t_{\mathrm{sim}}$ the system can still remain collisionless, and the orbits of the simulation particles will faithfully represent the motion of the fundamental particles

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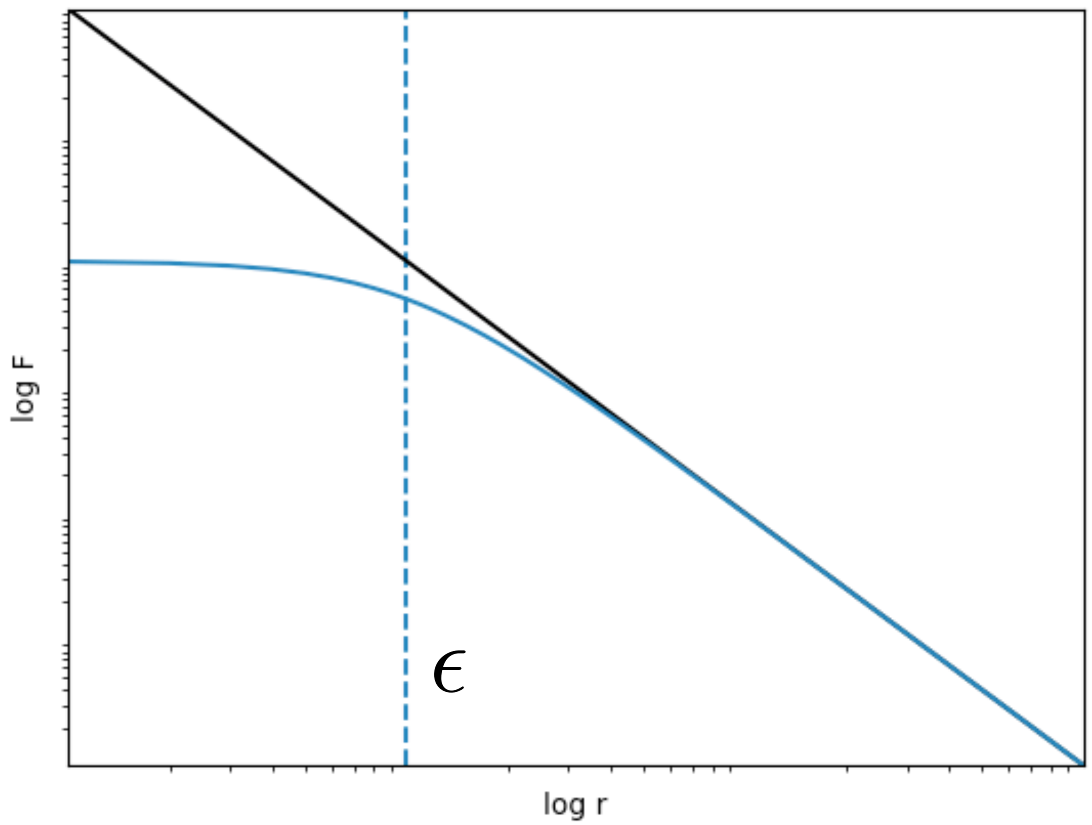

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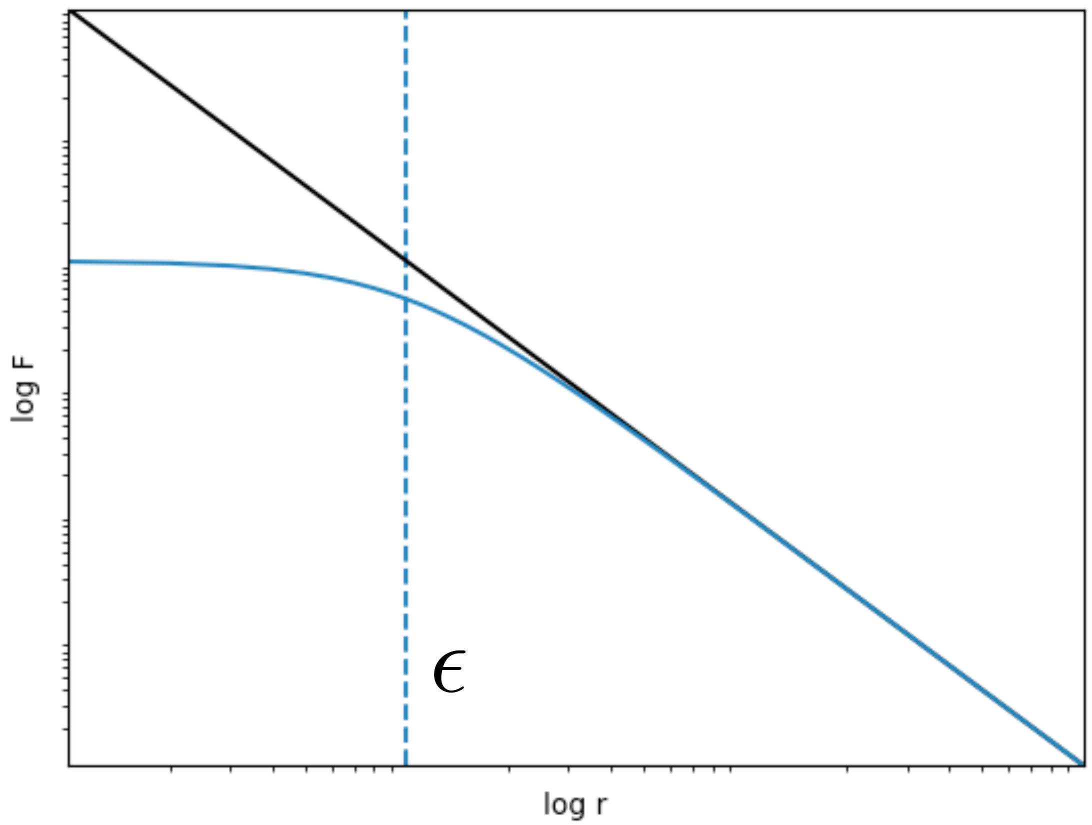

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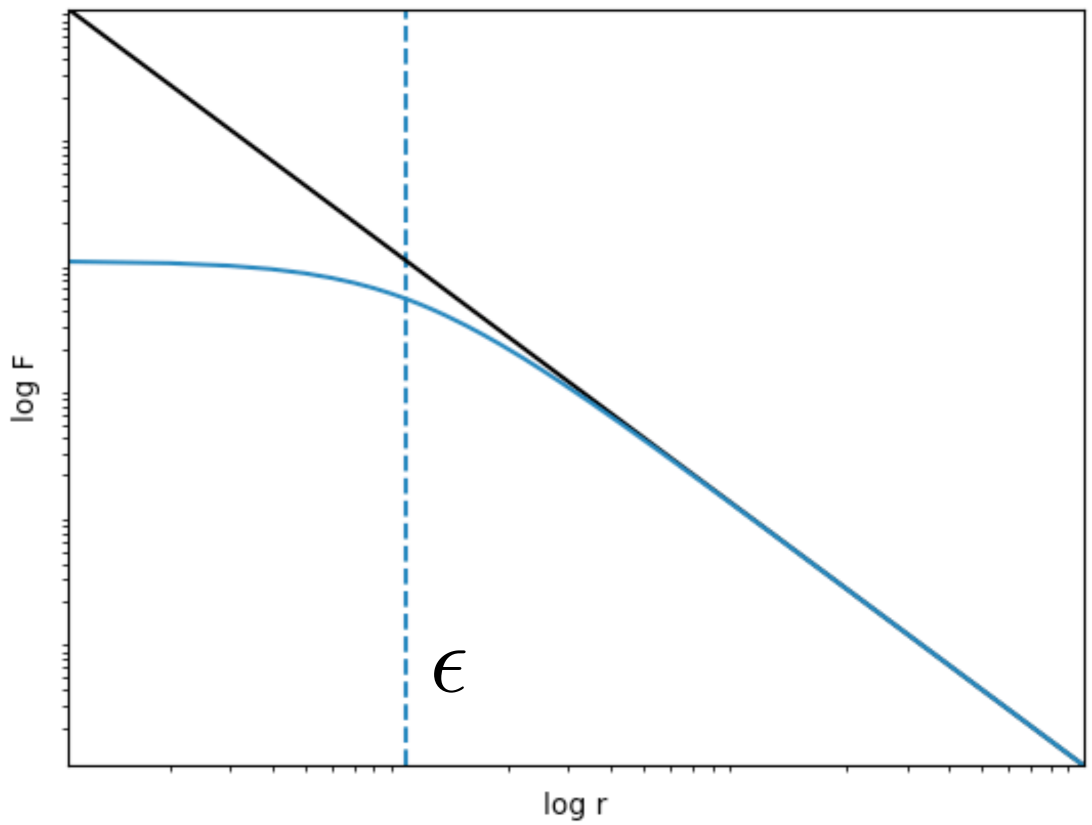

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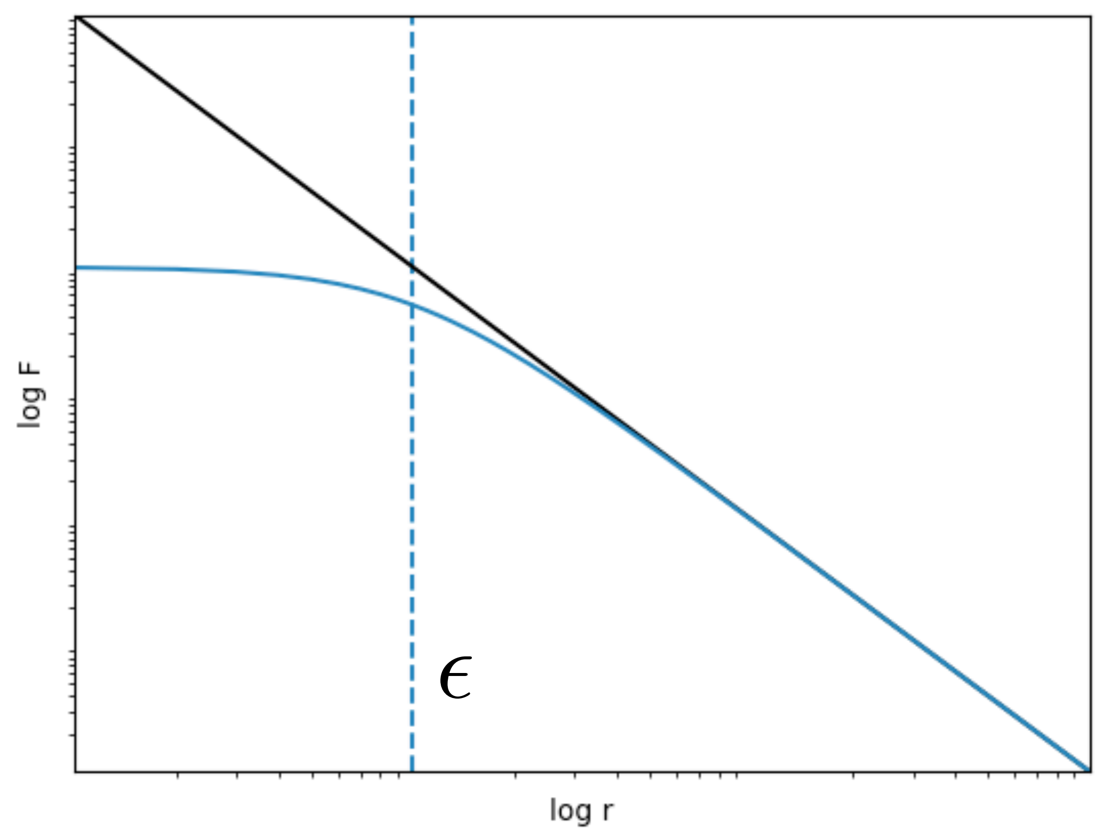

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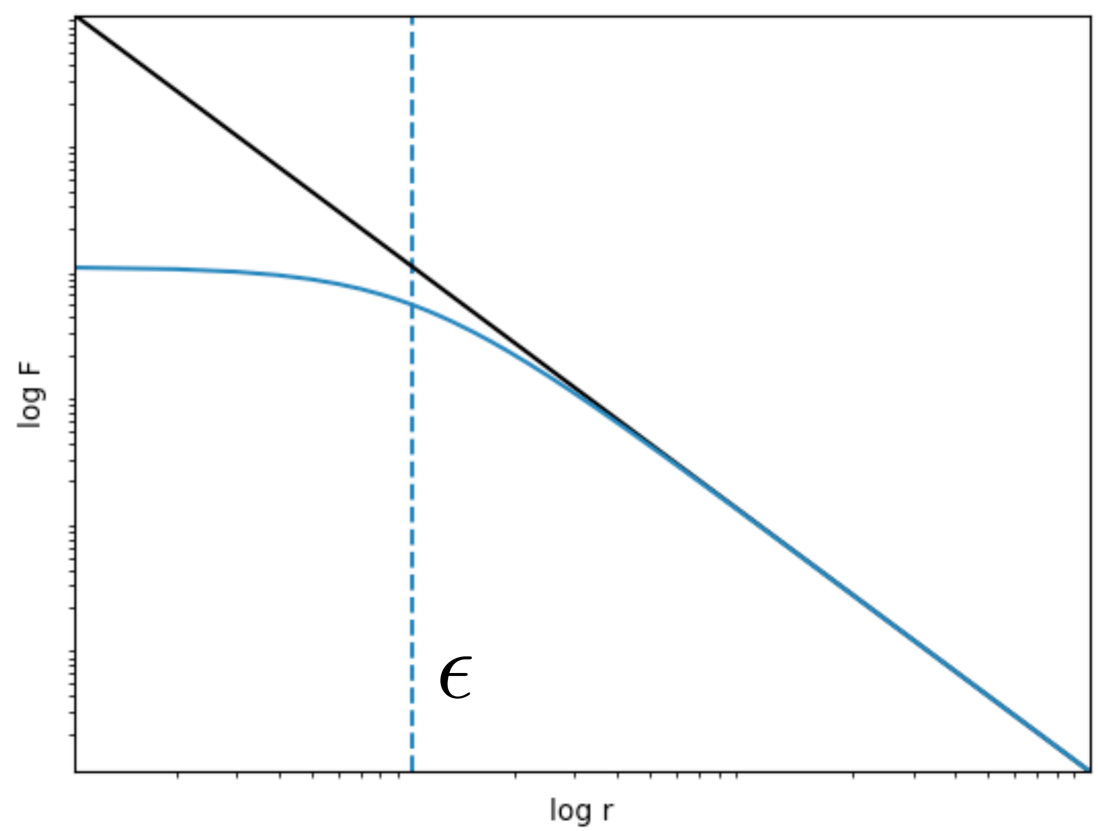

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ϵ typically 2-4% of the mean inter-particle separation

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for which we need to know the peculiar potential ϕ that follows:

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Solving gravity: Particle-Mesh

Solving gravity: the Particle-Mesh (PM) method

A possible way to solve for the gravitational potential is the so-called particle-mesh (PM) method, which exploits the properties of the Poisson's equation in Fourier space.

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Instead of computing ϕ by direct summation on all particles, the solution to the gravitational Poisson's Equation

$$\nabla_x^2 \Phi = 4\pi G a^2 \rho(\mathbf{x})$$

can be obtained through a **convolution of the density field** $\rho(\mathbf{x})$ **with a Green's Function**

$$\Phi(\mathbf{x}) = \int g(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}' = g(\mathbf{x}) \star \rho(\mathbf{x})$$

Solving gravity: the Particle-Mesh (PM) method

Example: for vacuum boundary conditions (i.e. the density goes to zero at infinity) one has:

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \;\Rightarrow\; g(\mathbf{x}) = -\frac{G}{|\mathbf{x}|}$$

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The PM method exploits the convolution theorem:

in Fourier space the convolution becomes a simple multiplication

(between the Fourier transform of the Green's Function g_k and the Fourier transform of the density field ρ_k) so that the gravitational potential in Fourier space Φ_k can be simply computed as:

$$\Phi_k = g_k \cdot \rho_k$$

Solving gravity: the Particle-Mesh (PM) method

Therefore, the PM method requires **three mathematical operations to solve for the gravitational potential**:

1. Compute the Fourier transform of the density field ρ_k
2. Multiply the density field in Fourier space with the Green's function to obtain the potential $\Phi_k = g_k \cdot \rho_k$
3. Compute the inverse Fourier transform of the potential to position space to get $\Phi_k \rightarrow \Phi(\mathbf{x})$

However, in a system of N discrete particles, one does not have a density field, which has to be computed starting from the particles.

Solving gravity: the Particle-Mesh (PM) method

Therefore, the PM algorithm is made of four different steps, only one of which is the actual potential computation:

A. **Density assignment** (from particle's position to a density field)
B. **Potential computation** (comprising the 3 mathematical operations discussed above)
C. **Computation of the force** (acceleration) field from the potential
D. **Assign the forces to individual particles**

Solving gravity: PM mass assignment

Step A: Density assignment

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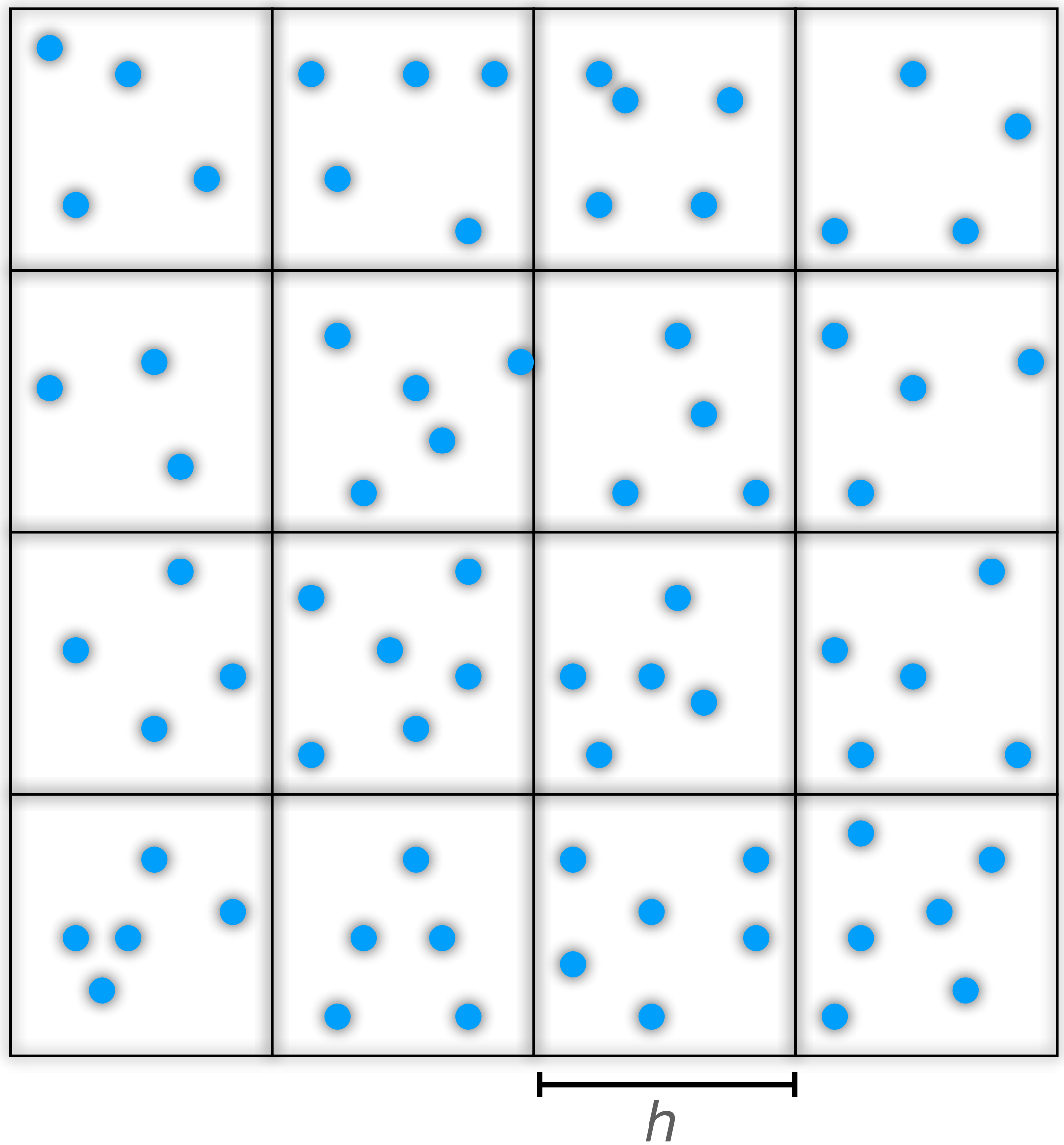

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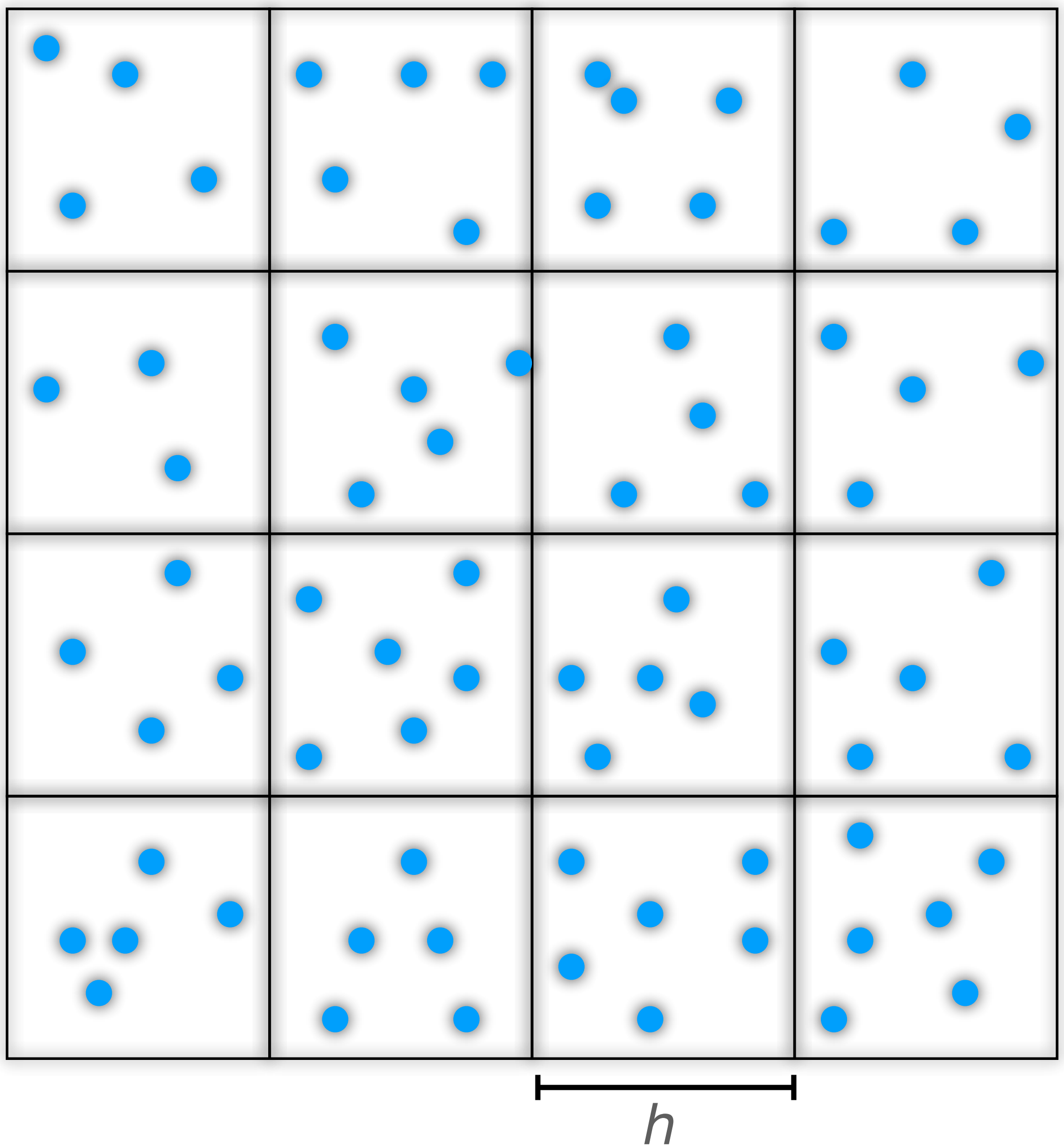


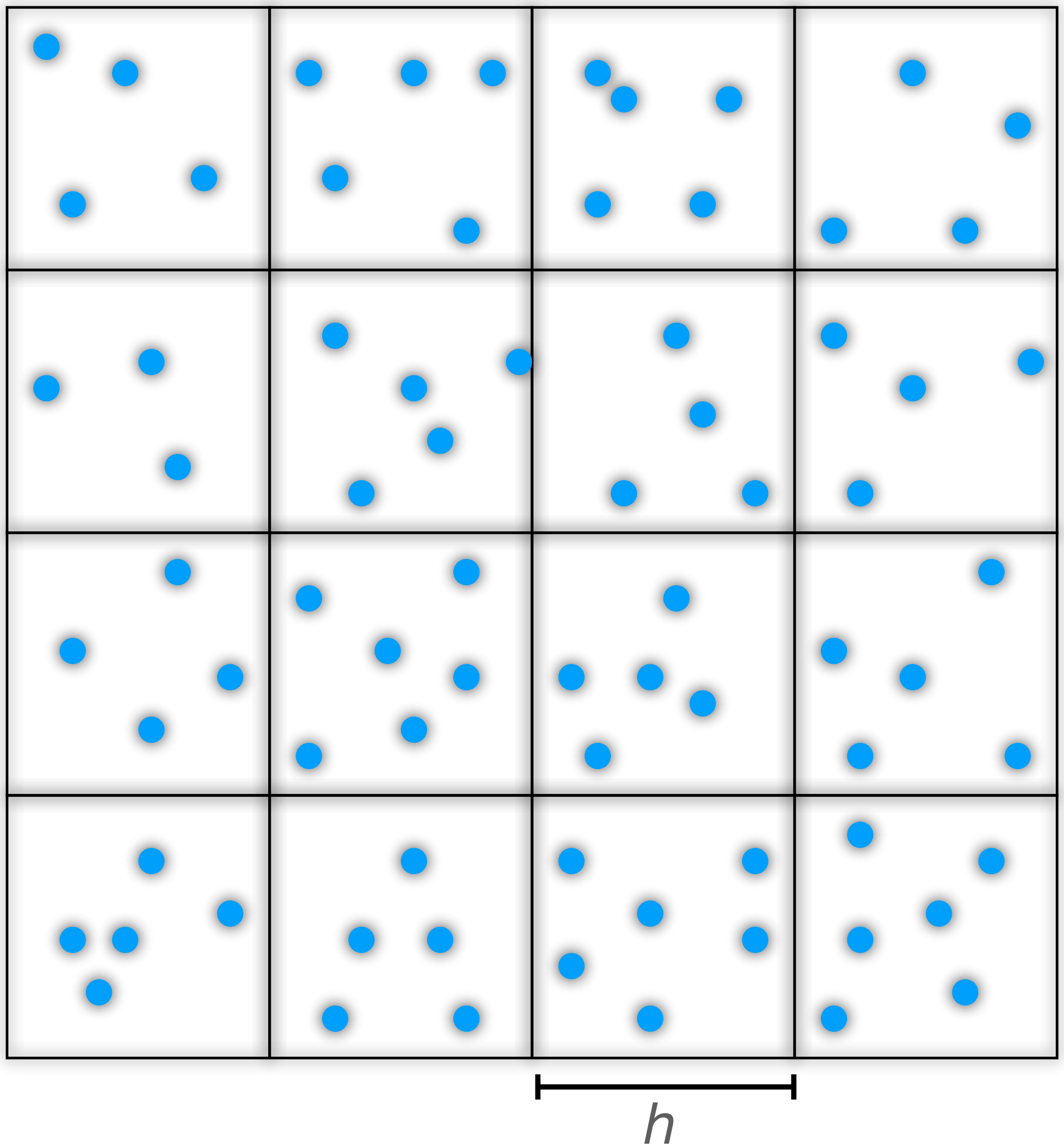
We can give particles a "shape" *S(x)* with a corresponding volume of uniform density, and assign to each mesh cell the fraction of the particle's mass that falls inside the cell.

If we call $\mathbf{x_m}$ the position of the cell centres and $\mathbf{x}_i$ the position of particle *i* the overlap of a cell (i.e. the fraction of the particle's volume falling in the cell) is given by:

Solving gravity: PM mass assignment

Step A: Density assignment

$$W(\mathbf{x}_\mathrm{m} - \mathbf{x}_i) = \int_{\mathbf{x}_\mathrm{m}-\frac{h}{2}}^{\mathbf{x}_\mathrm{m}+\frac{h}{2}} S(\mathbf{x}' - \mathbf{x}_i)\, d\mathbf{x}'$$



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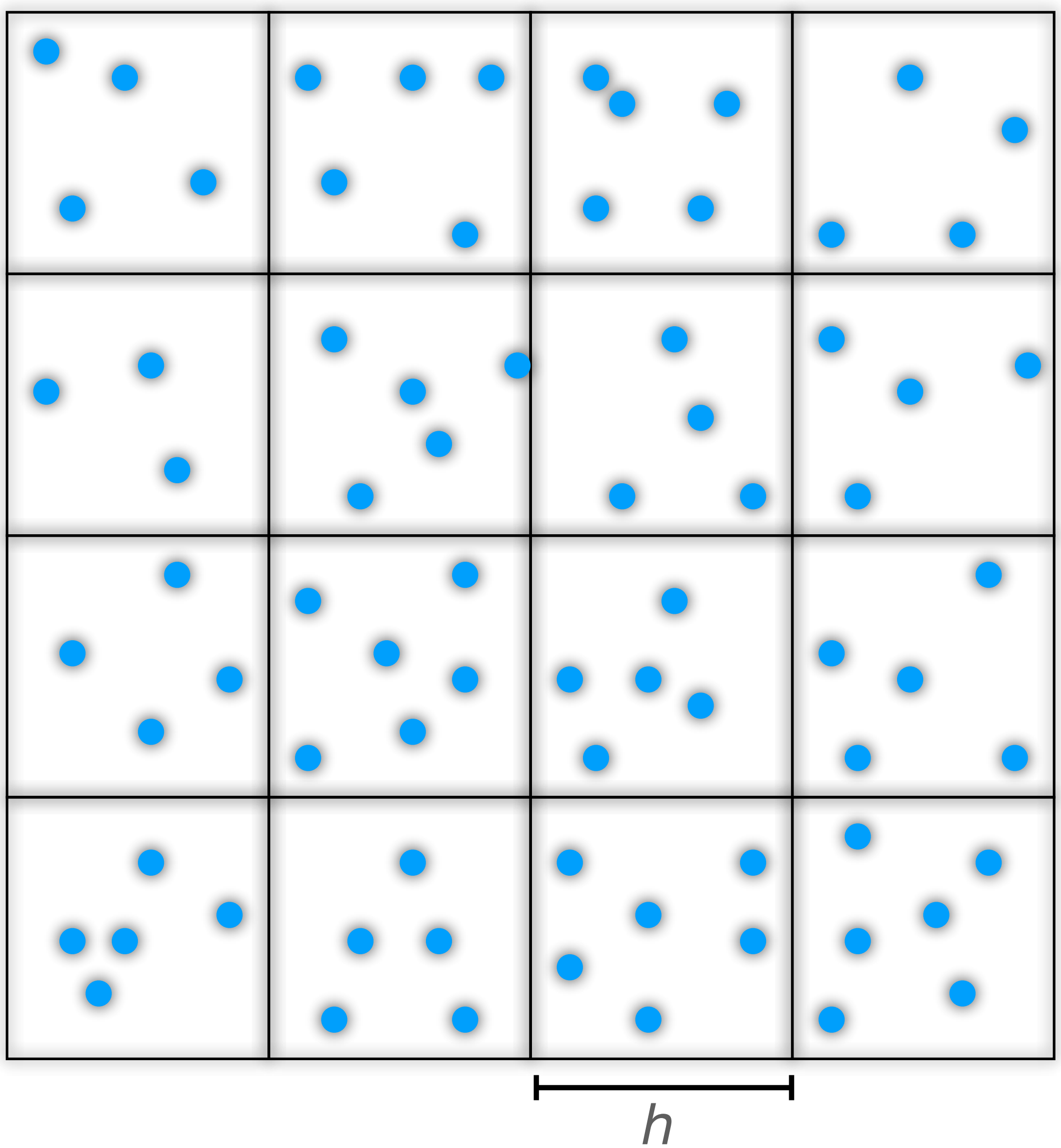

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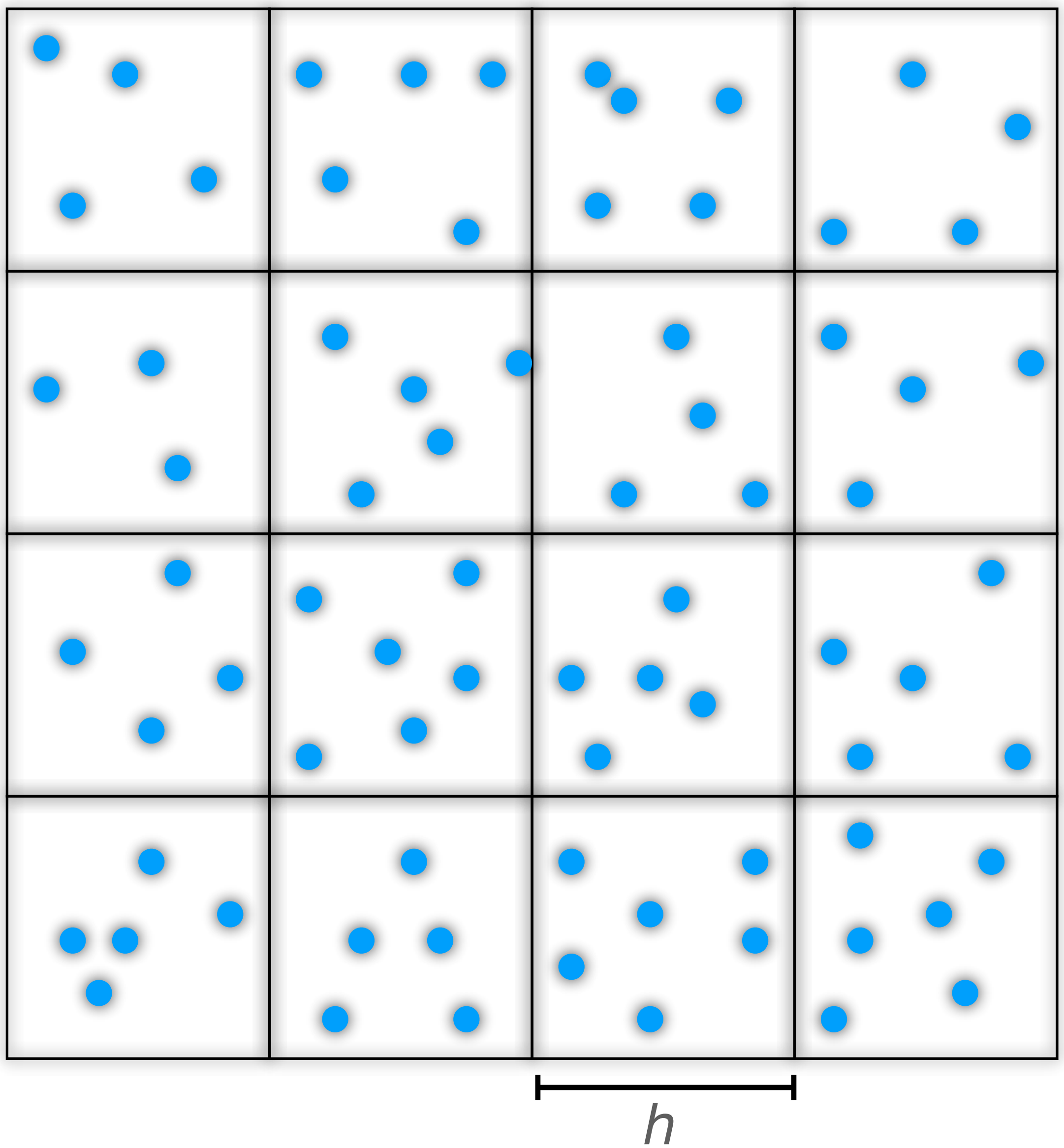

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So the density assignment function *W* is the convolution

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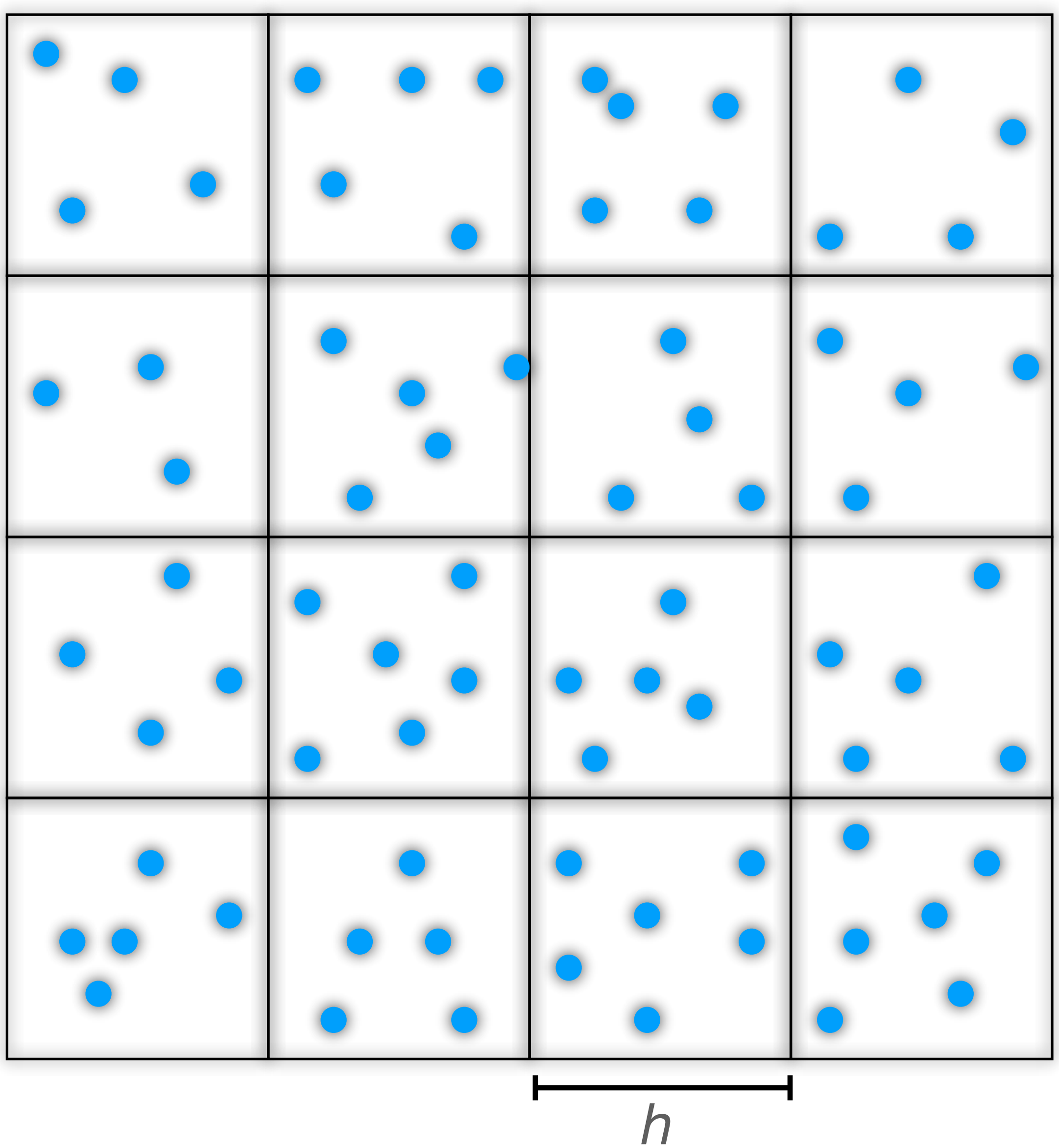
Step A: Density assignment

$$W(\mathbf{x}_\mathrm{m} - \mathbf{x}_i) = \int_{\mathbf{x}_\mathrm{m}-\frac{h}{2}}^{\mathbf{x}_\mathrm{m}+\frac{h}{2}} S(\mathbf{x}' - \mathbf{x}_i)d\mathbf{x}' = \int \Pi\left(\frac{\mathbf{x}' - \mathbf{x}_\mathrm{m}}{h}\right) S(\mathbf{x}' - \mathbf{x}_i)d\mathbf{x}'$$



where we have defined:

$$\Pi(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

So the density assignment function *W* is the convolution

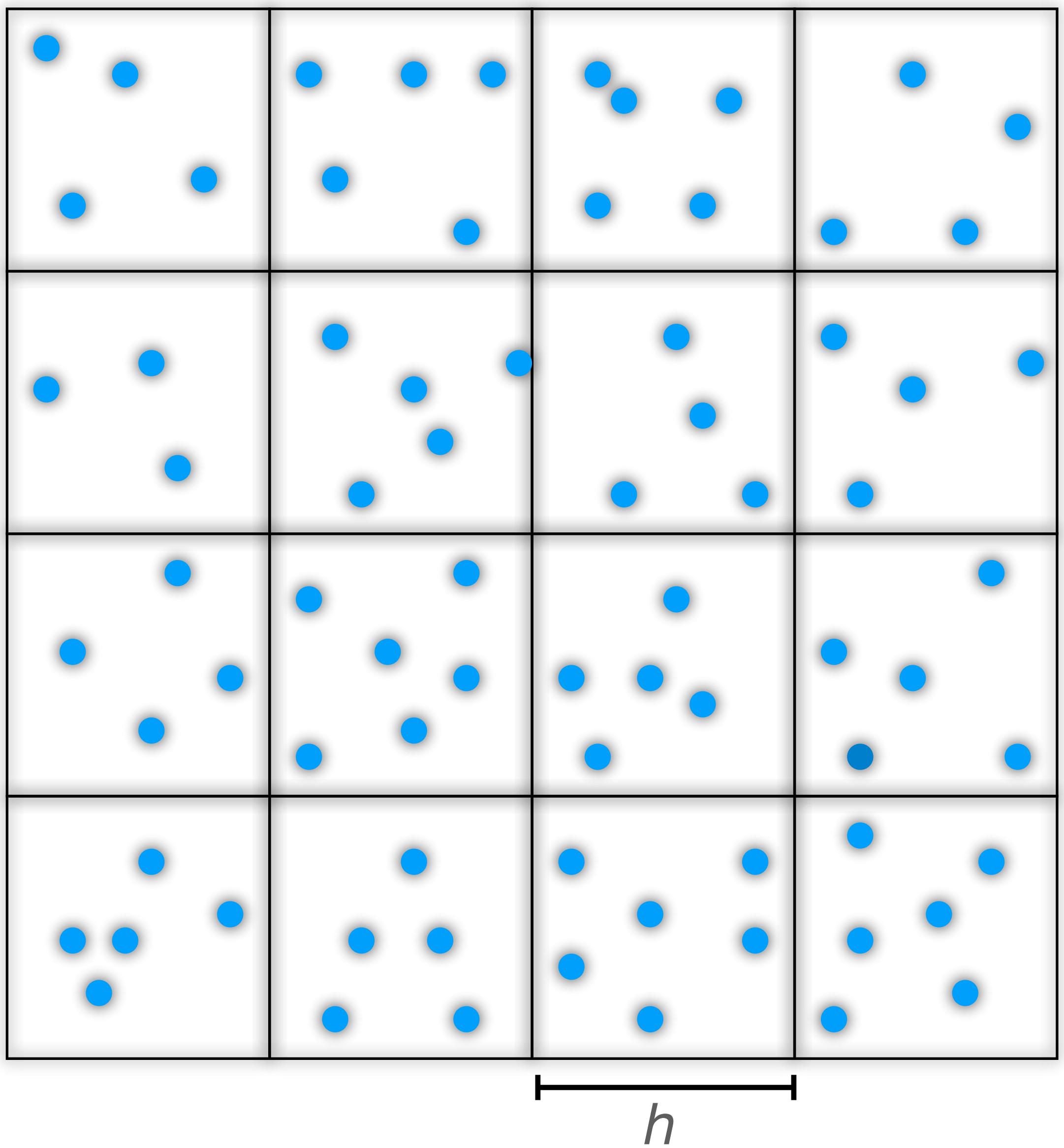
$$W(\mathbf{x}) = \Pi\left(\frac{\mathbf{x}}{h}\right) \star S(\mathbf{x})$$

and the density on the grid is:

$$\rho(\mathbf{x}_\mathrm{m}) = \frac{1}{h^3}\sum_{i=1}^{N} m_i W(\mathbf{x}_i - \mathbf{x}_\mathrm{m})$$

Solving gravity: PM mass assignment

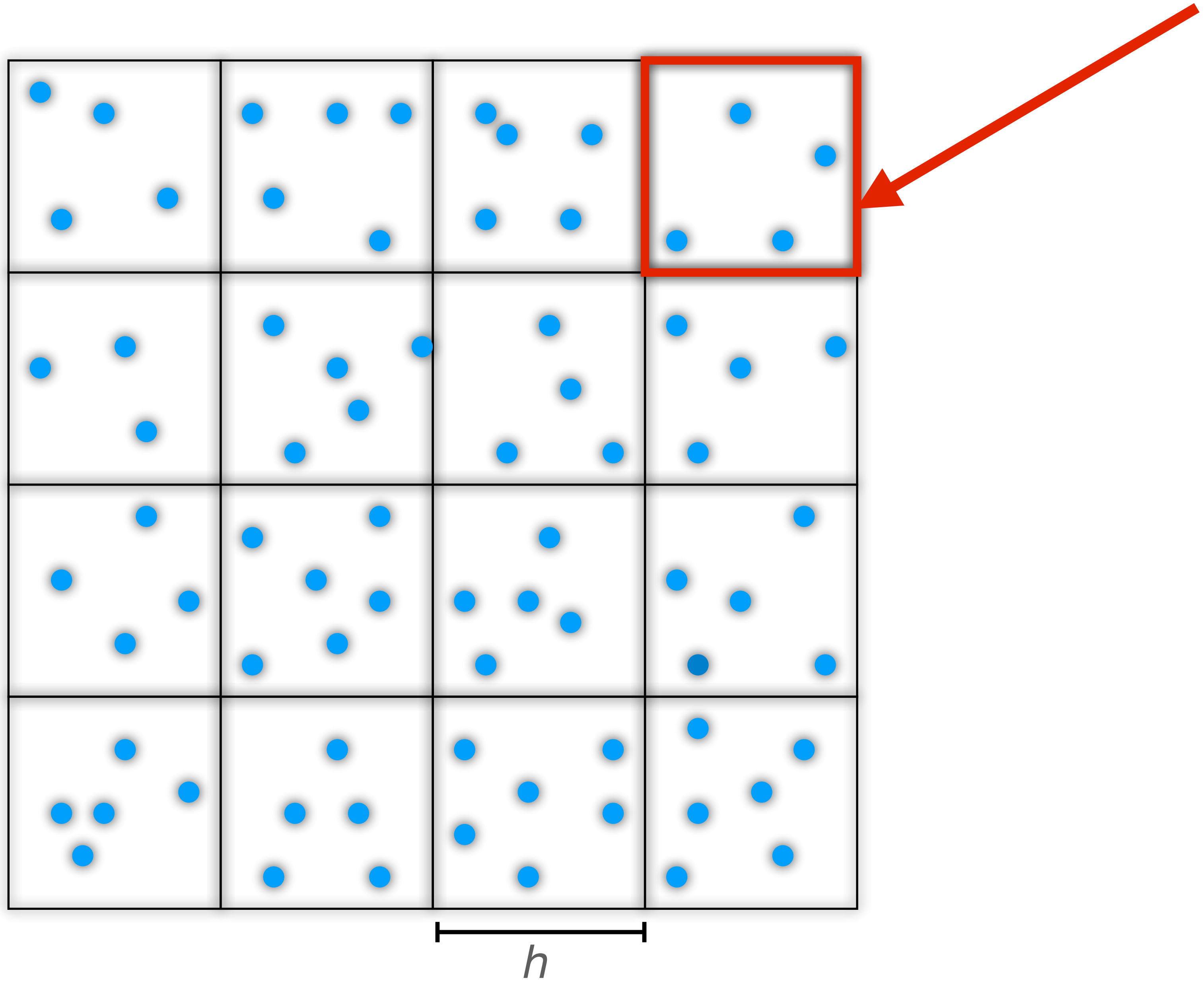
Step A: Density assignment



Solving gravity: PM mass assignment

Step A: Density assignment

Shape function: a Dirac delta
A single particle contributes all its mass to the cell it belongs to
Density in the cell $\rho = 4\ m\ /\ h^3$



Solving gravity: PM mass assignment

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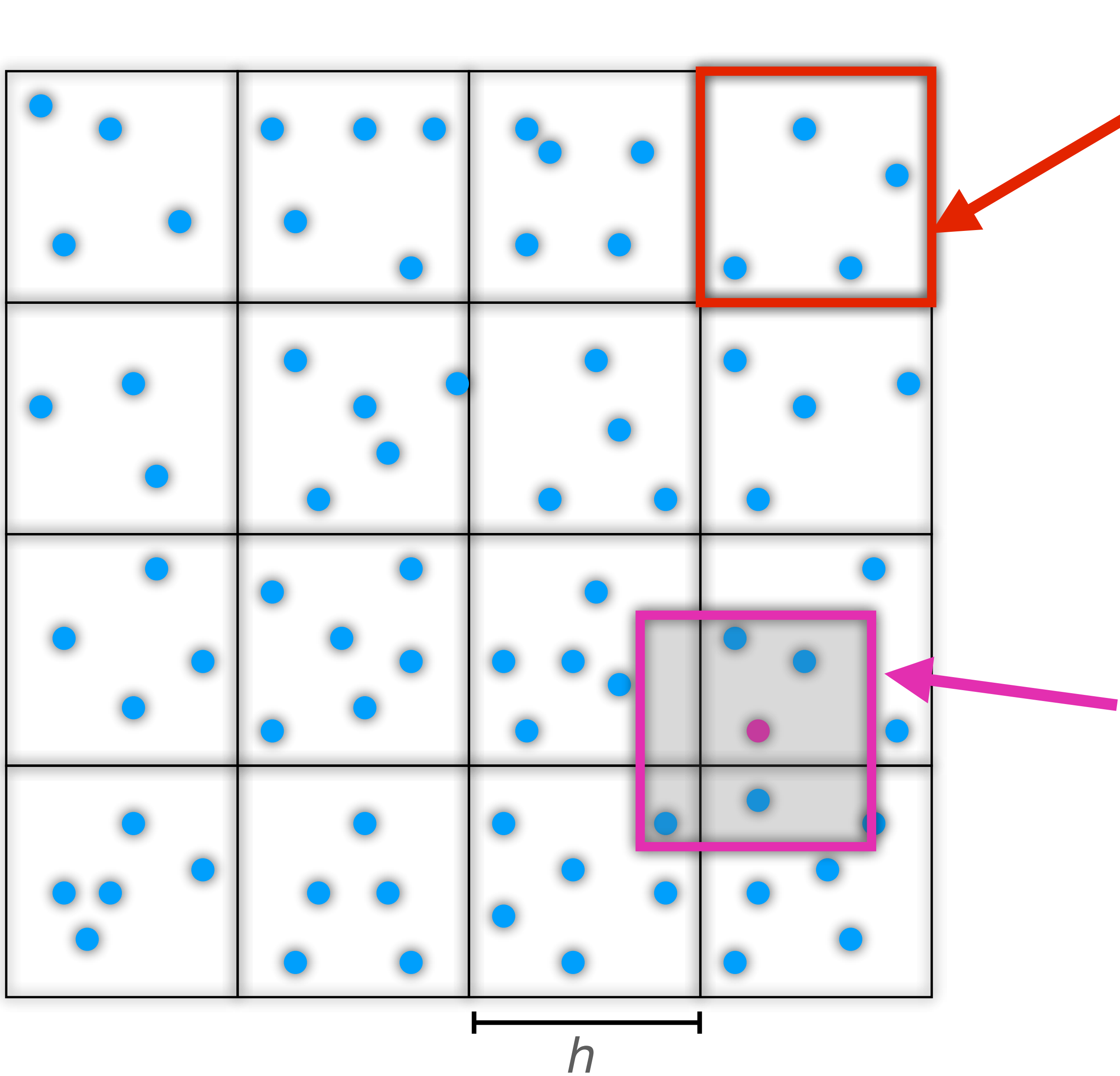

Shape function: a cubic volume of side *h*
A single particle contributes some mass to 8 different grid cells

Solving gravity: PM mass assignment

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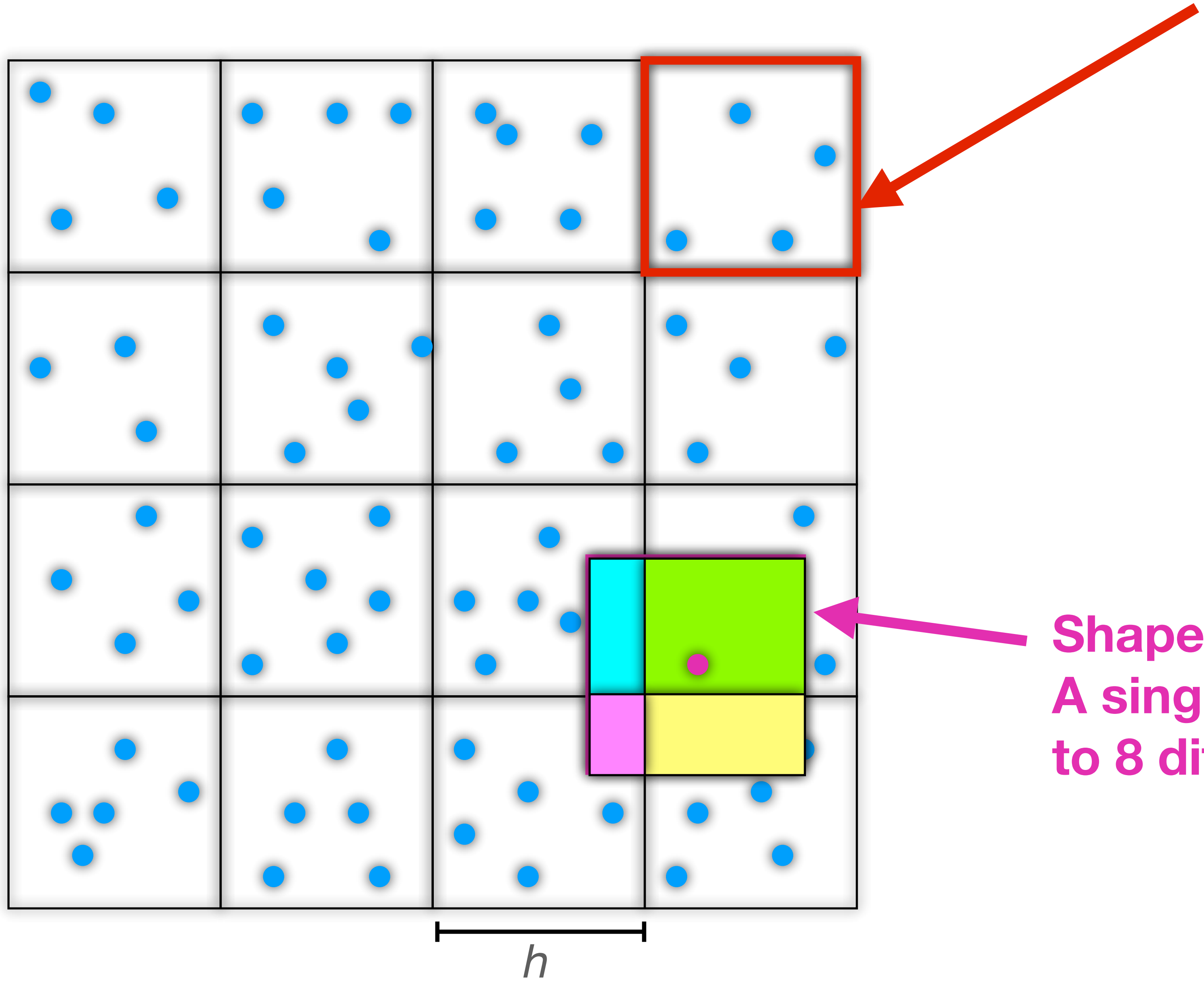

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Solving gravity: PM mass assignment

Step A: Density assignment

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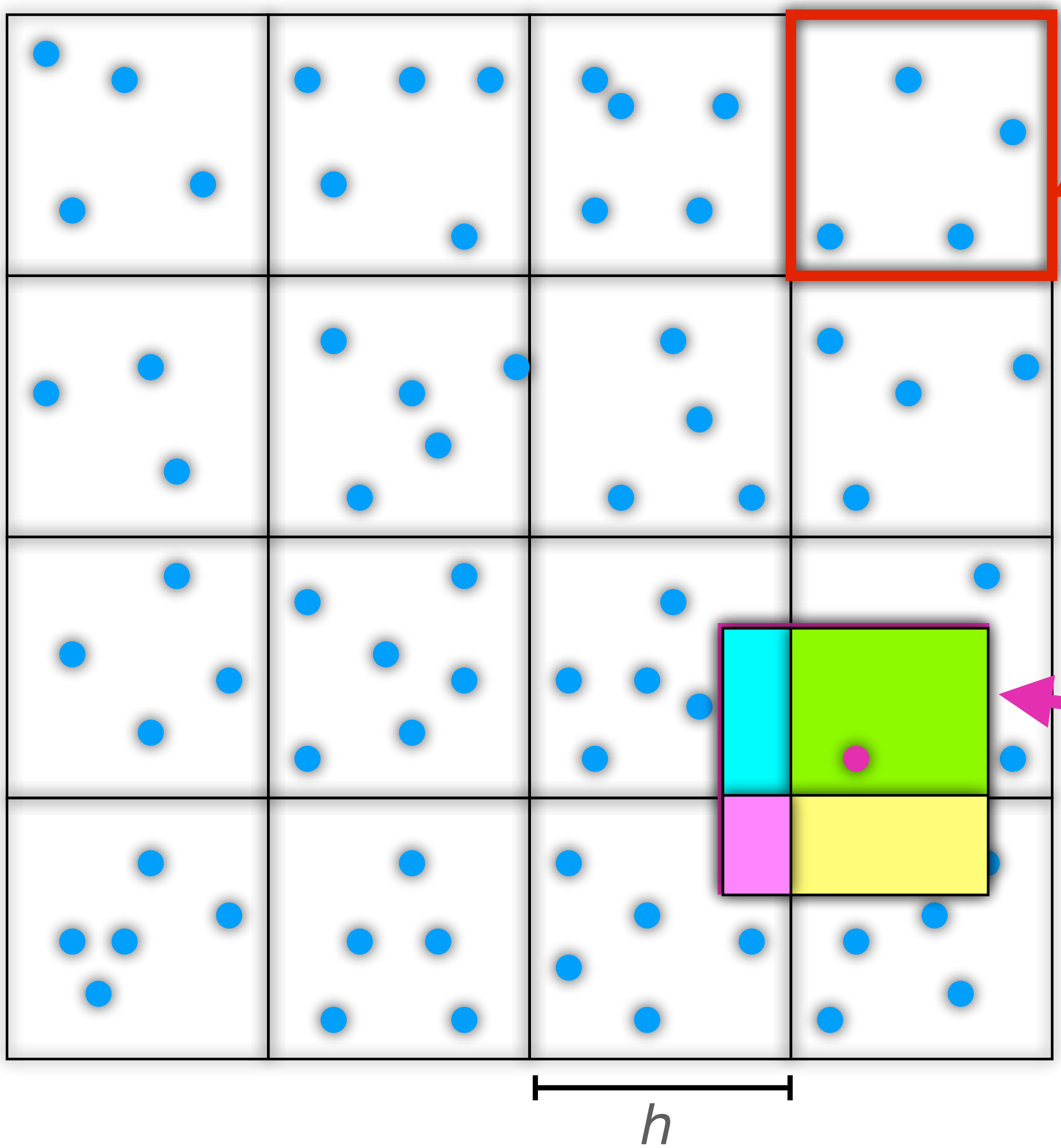
This is called Nearest-Grid-Point, or NGP



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Solving gravity: PM mass assignment

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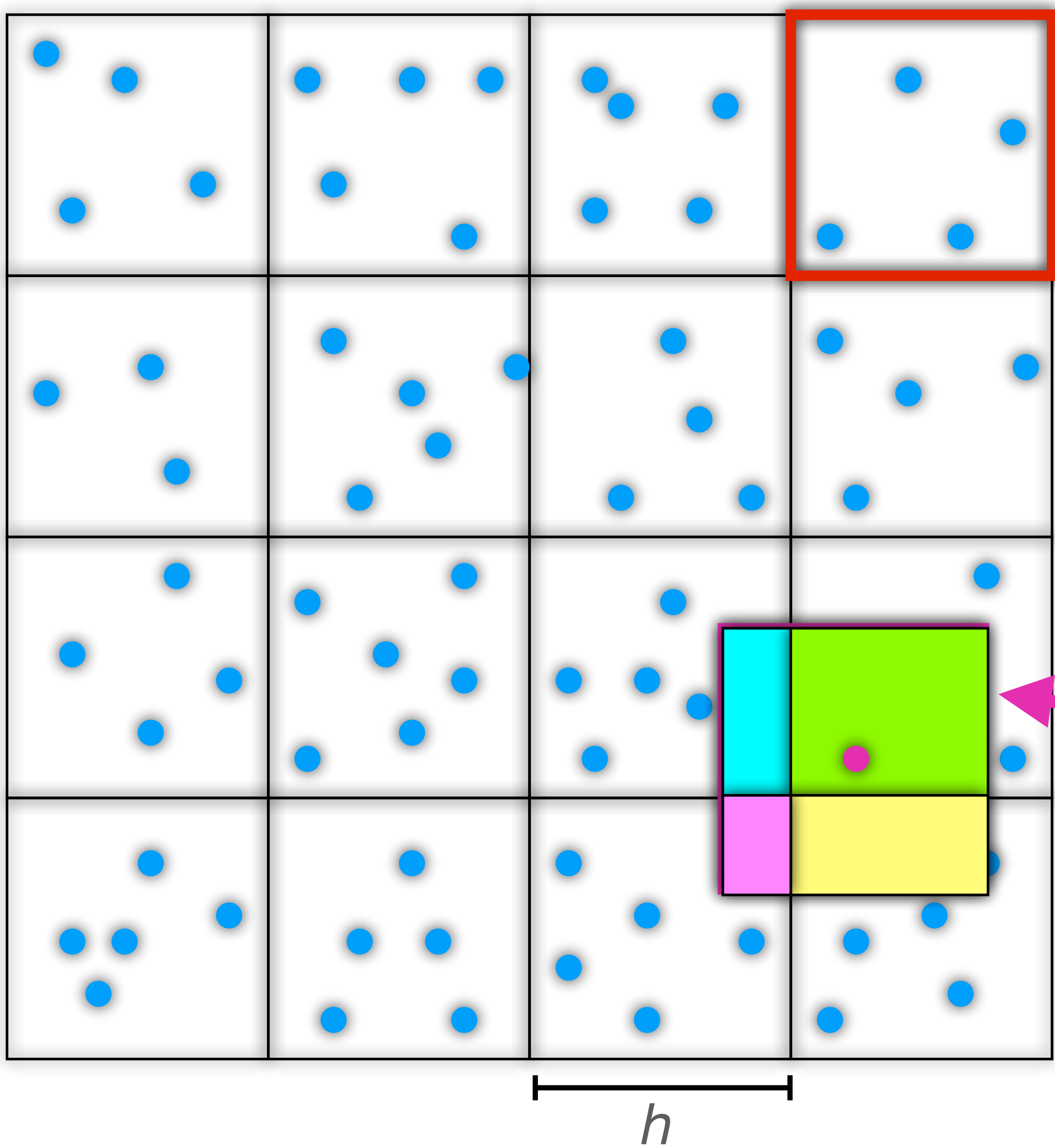

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Shape function: a cubic volume of side *h*
A single particle contributes some mass to 8 different grid cells

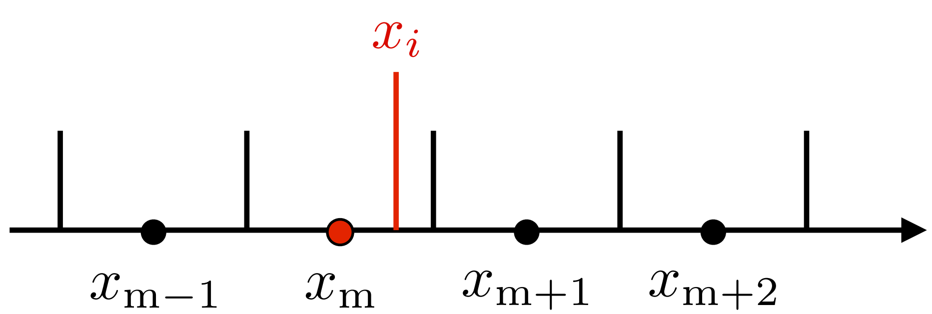
This is called Cloud-In-Cell, or CIC

Solving gravity: PM mass assignment

Step A: Density assignment

Some popular Shape Functions for the density assignment

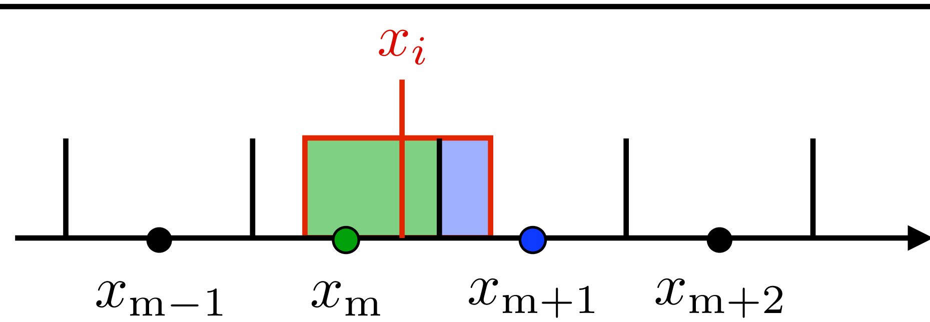
Nearest Grid Point
NGP



Mass is distributed over D^1 cells

Resulting density (and force) piecewise constant

Clouds-In-Cell
CIC

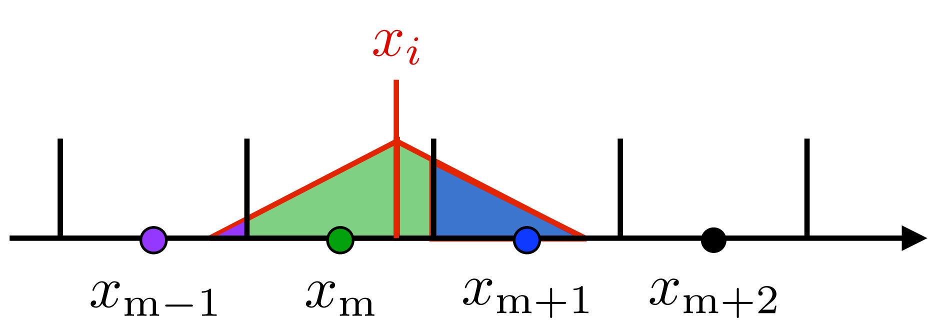


Mass is distributed over D^2 cells

Resulting density (and force) piecewise linear and continuous

---

Triangular Shaped Cloud
TSC



Mass is distributed over D^3 cells

Resulting density (and force) have continuous first derivative

---

For the force assignment (step D), the same assignment function used for step A needs to be used **to ensure momentum conservation**.

Solving gravity: PM potential computation

Step B: Potential Computation

Once the density field in real space has been obtained, in order to compute the gravitational potential **we first need to get its Fourier transform**. If we assume a **periodic** density field in a cubic box of size L, this can be expressed as a Fourier series:

$$\rho(\mathbf{x}) = \sum_{\mathbf{k}} \rho_k e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{where} \qquad \mathbf{k} \in \frac{2\pi}{L}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Solving gravity: PM potential computation

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One can do the same thing for the potential

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \Phi_k e^{i\mathbf{k}\cdot\mathbf{x}}$$

Solving gravity: PM potential computation

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One can do the same thing for the potential

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \Phi_k e^{i\mathbf{k}\cdot\mathbf{x}}$$

and after substitution, the Poisson equation reads:

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}) \;\Rightarrow\; \nabla^2\left(\sum_{\mathbf{k}} \Phi_k e^{i\mathbf{k}\cdot\mathbf{x}}\right) = 4\pi G\left(\sum_{\mathbf{k}} \rho_k e^{i\mathbf{k}\cdot\mathbf{x}}\right)$$

Solving gravity: PM potential computation

Step B: Potential Computation

The derivative on the LHS can be easily performed:

$$\sum_{\mathbf{k}} \left(-k^2 \Phi_k\right) e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi G \sum_{\mathbf{k}} \rho_k e^{i\mathbf{k}\cdot\mathbf{x}}$$

Solving gravity: PM potential computation

Step B: Potential Computation

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As the equality must hold for each Fourier mode k, this gives the solution for the gravitational potential in Fourier space:

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Solving gravity: PM potential computation

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From which we see that the Green's function for the Poisson's equation with periodic boundary conditions is given by:

$$g_k = -\frac{4\pi G}{k^2}$$

Solving gravity: PM potential computation

Step B: Potential Computation

We can now Fourier transform back the gravitational potential to real space, and have a potential field defined in each of the grid cells.

$$\mathcal{F}^{-1}(\Phi_k) = \Phi(\mathbf{x}_{\rm m})$$

We can now compute the forces at the cells' centers…

Solving gravity: PM force computation

Step C: Force Computation

Once the gravitational potential in real space is known, one has to compute the force field in each of the grid cells. This is done by approximating the force field by finite differencing:

$$\mathbf{f} = -\nabla(\Phi)$$

Solving gravity: PM force computation

Step C: Force Computation

Once the gravitational potential in real space is known, one has to compute the force field in each of the grid cells. This is done by approximating the force field by finite differencing:

$$\mathbf{f} = -\nabla(\Phi)$$

Possible finite differencing schemes:

2nd order:

$$f^{(x)}_{i,j,k} = -\frac{\Phi_{i+1,j,k} - \Phi_{i-1,j,k}}{2h}$$

Solving gravity: PM force computation

Step C: Force Computation

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2nd order:

$$f^{(x)}_{i,j,k} = -\frac{\Phi_{i+1,j,k} - \Phi_{i-1,j,k}}{2h}$$

4th order:

$$f^{(x)}_{i,j,k} = -\frac{4}{3}\frac{\Phi_{i+1,j,k} - \Phi_{i-1,j,k}}{2h} + \frac{1}{3}\frac{\Phi_{i+2,j,k} - \Phi_{i-2,j,k}}{4h}$$

Solving gravity: PM force computation

Step D: Force assignment

Once the force field on the grid cells in real space is known, in order to evolve the particles system one needs to compute the forces (or the accelerations) on the particles' positions.

Solving gravity: PM force computation

Step D: Force assignment

Once the force field on the grid cells in real space is known, in order to evolve the particles system one needs to compute the forces (or the accelerations) on the particles' positions.

This is done by interpolation:

$$F(\mathbf{x}_i) = \sum_{\mathbf{m}} W(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})\, f_{\mathbf{m}}$$

where **the same interpolation kernel W that was used for the density assignment in Step A MUST be used** to ensure for momentum conservation

Solving gravity: the Particle-Mesh method

Advantages and disadvantages of the PM method

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The main **advantage** of the PM method is that it is FAST and SIMPLE, complexity scales like N

Solving gravity: the Particle-Mesh method

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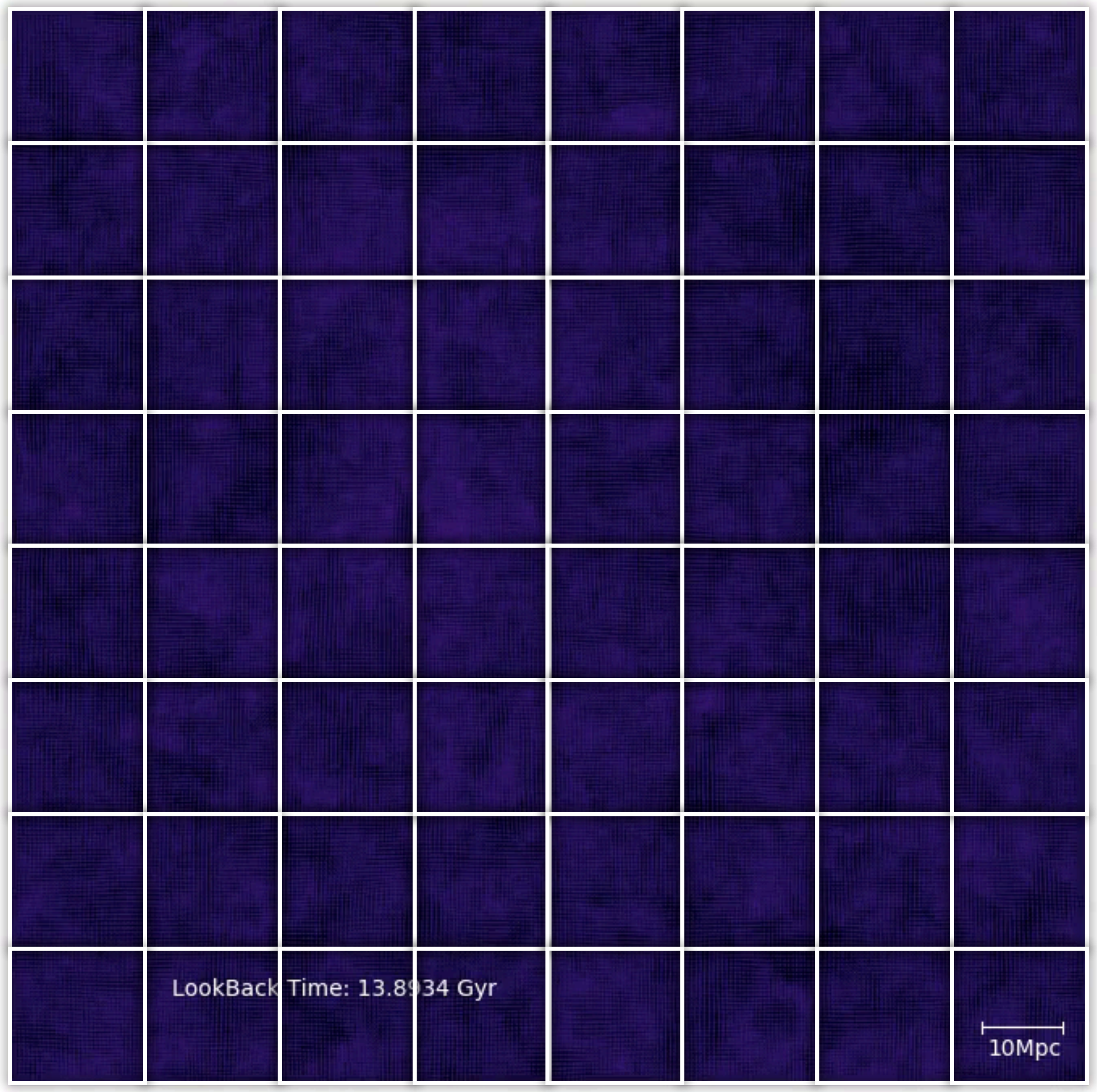
The main **disadvantage** of the PM algorithm is the fact that the **spatial resolution is limited to the mesh size** (h in our examples).

This is a very serious problem for astrophysical and (most importantly) for cosmological simulations, where the dynamic range of the problem is large: systems of interest may be unresolved as they cluster below the mesh scale

Solving gravity: the Particle-Mesh method

Advantages and disadvantages of the PM method



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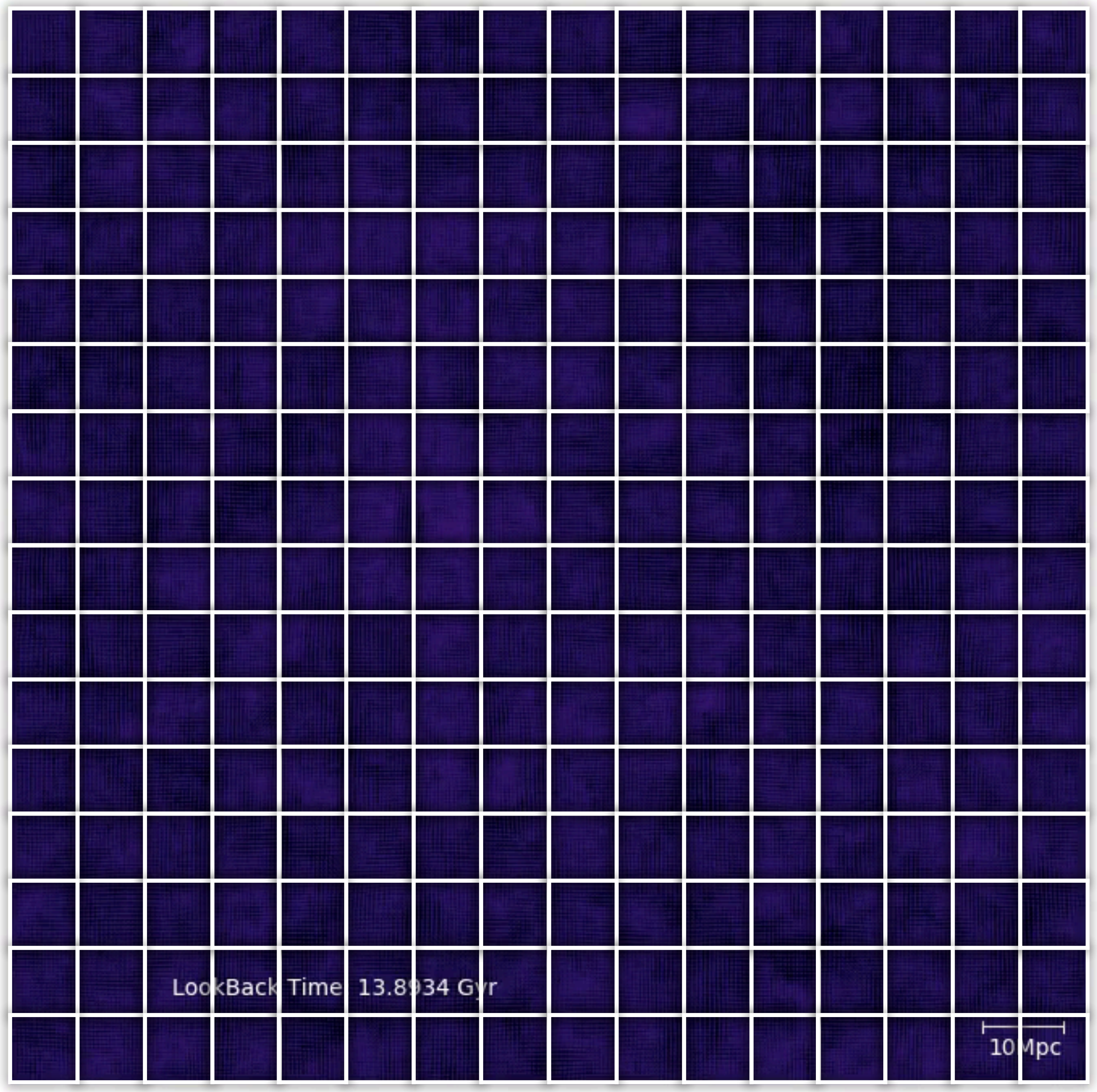

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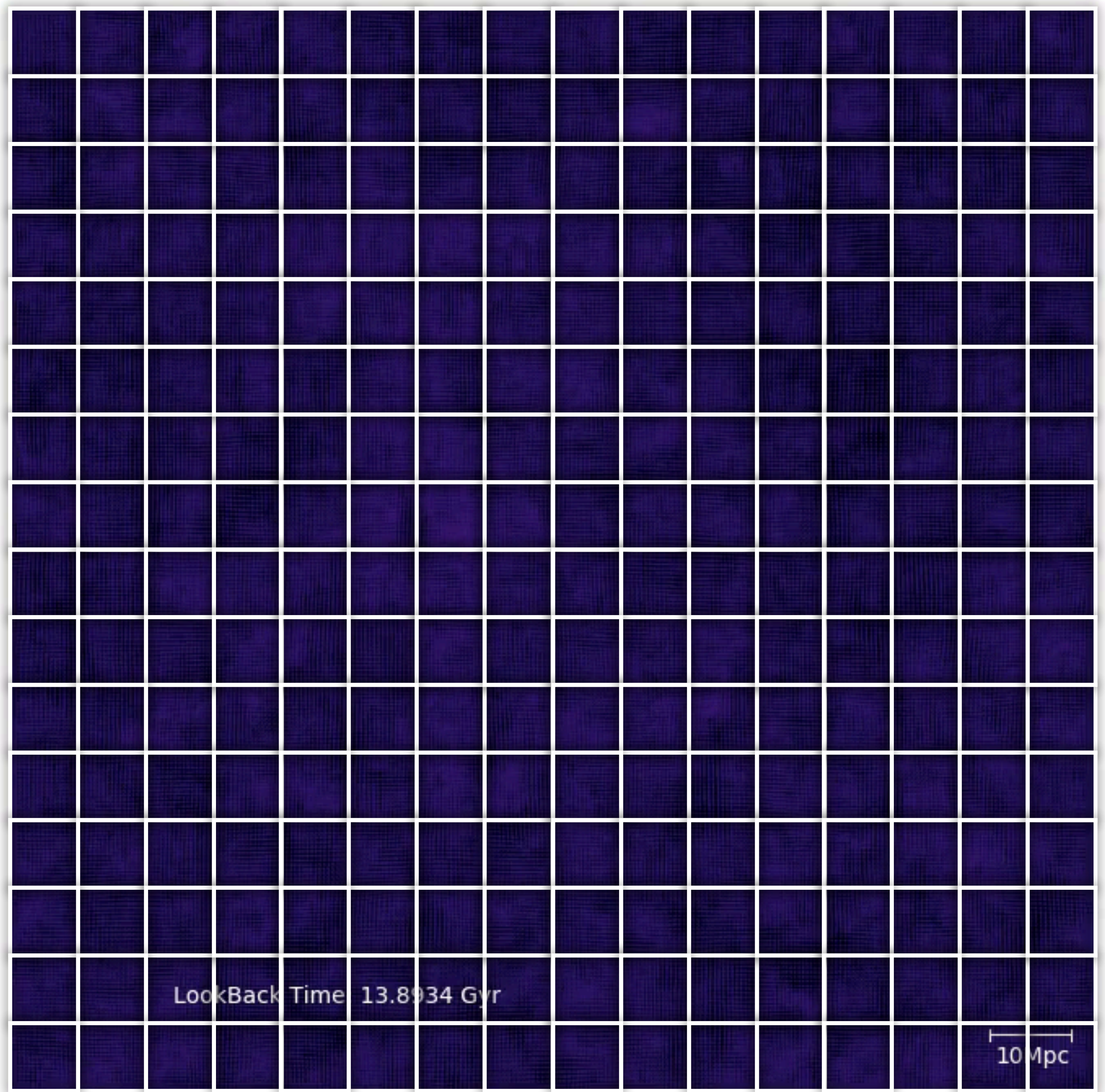


How?

Solving gravity: Tree methods

Solving gravity: the Tree method

A different method to solve the N-body problem is the so-called **Tree algorithm**: the simulation domain is recursively divided into sub-domains (tree nodes) forming the different levels of a hierarchical tree structure.

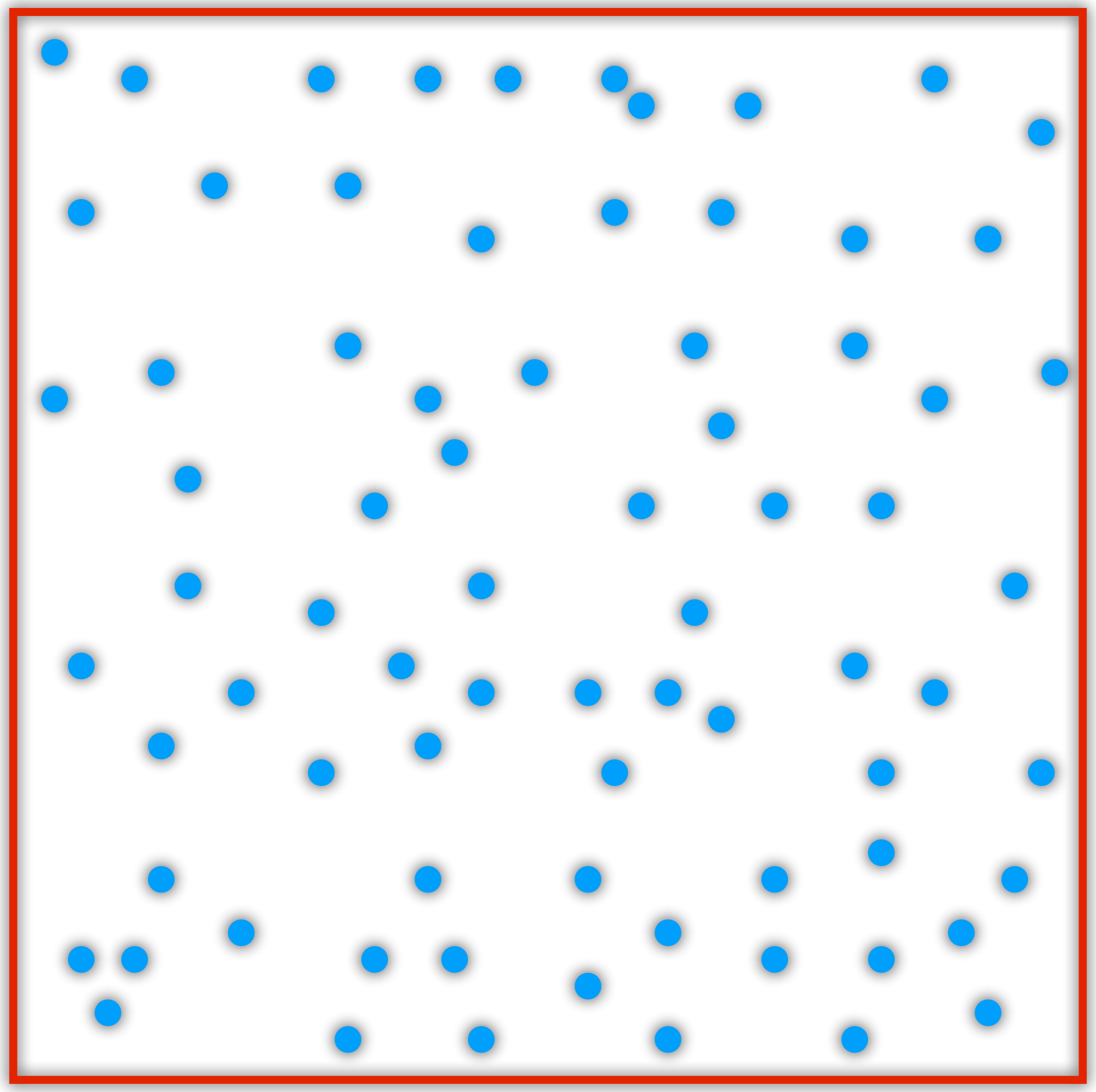
Solving gravity: the Tree method

A different method to solve the N-body problem is the so-called **Tree algorithm**: the simulation domain is recursively divided into sub-domains (tree nodes) forming the different levels of a hierarchical tree structure.

The main goal of such procedure is to **group distant particles together in the potential calculations and approximate their gravitational potential with a multipole expansion**. This reduces the complexity of the algorithm to O(N log[N]).

Solving gravity: the Tree method

Basic structure of a hierarchical tree:



ROOT node
(the entire simulation domain)

Solving gravity: the Tree method

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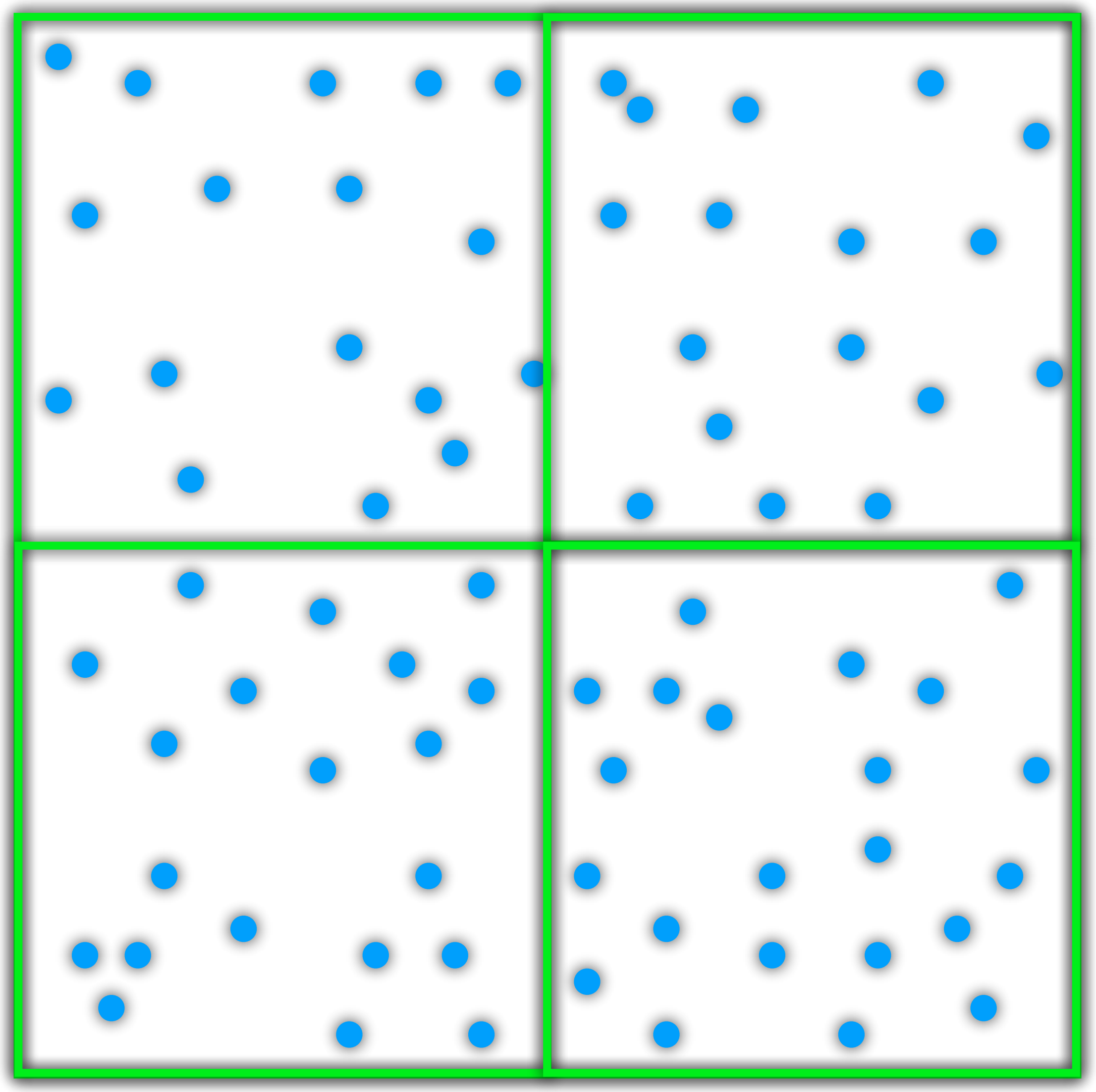

CHILD nodes - level 1
(1/8th of the simulation domain)

Solving gravity: the Tree method

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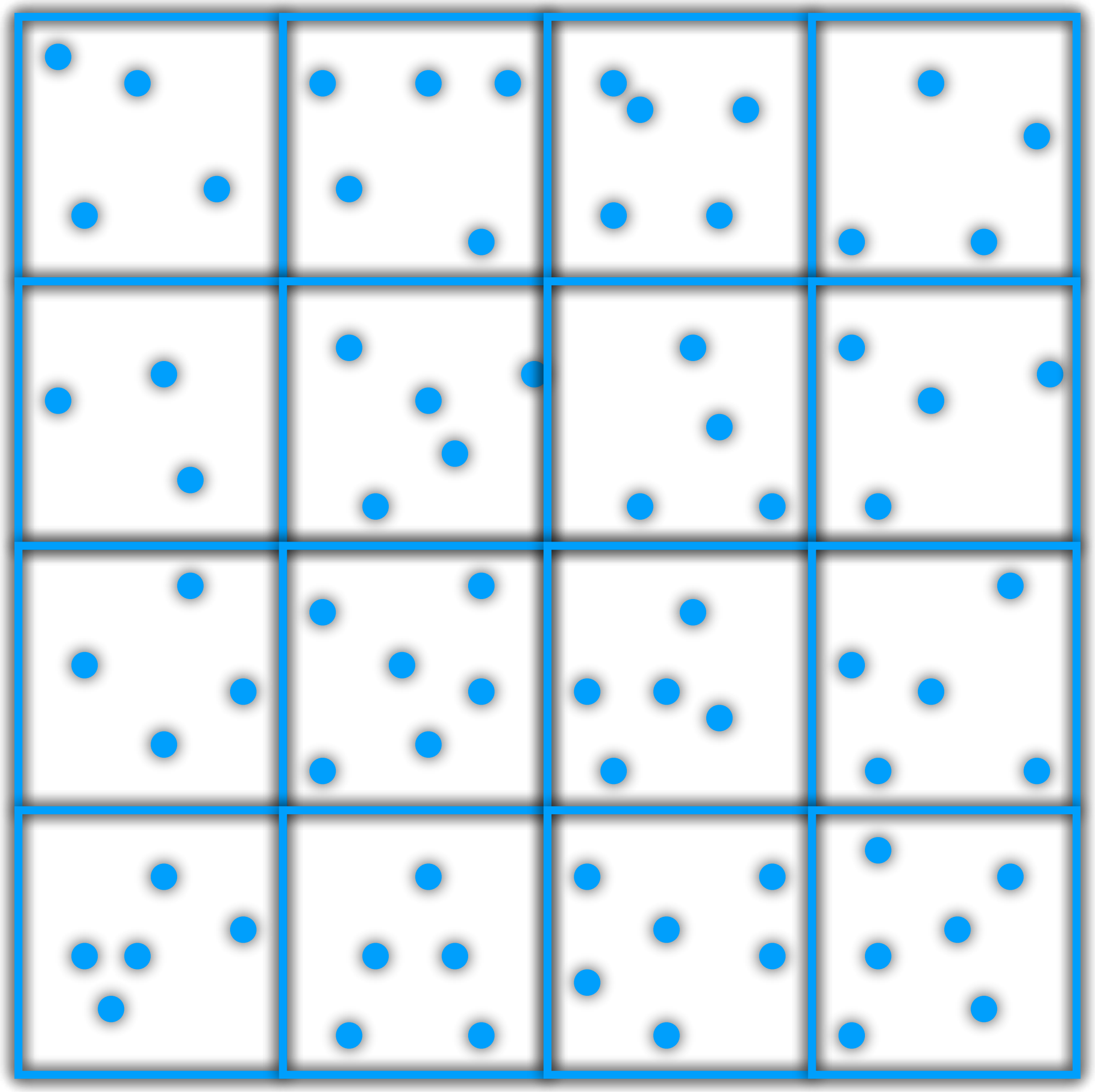


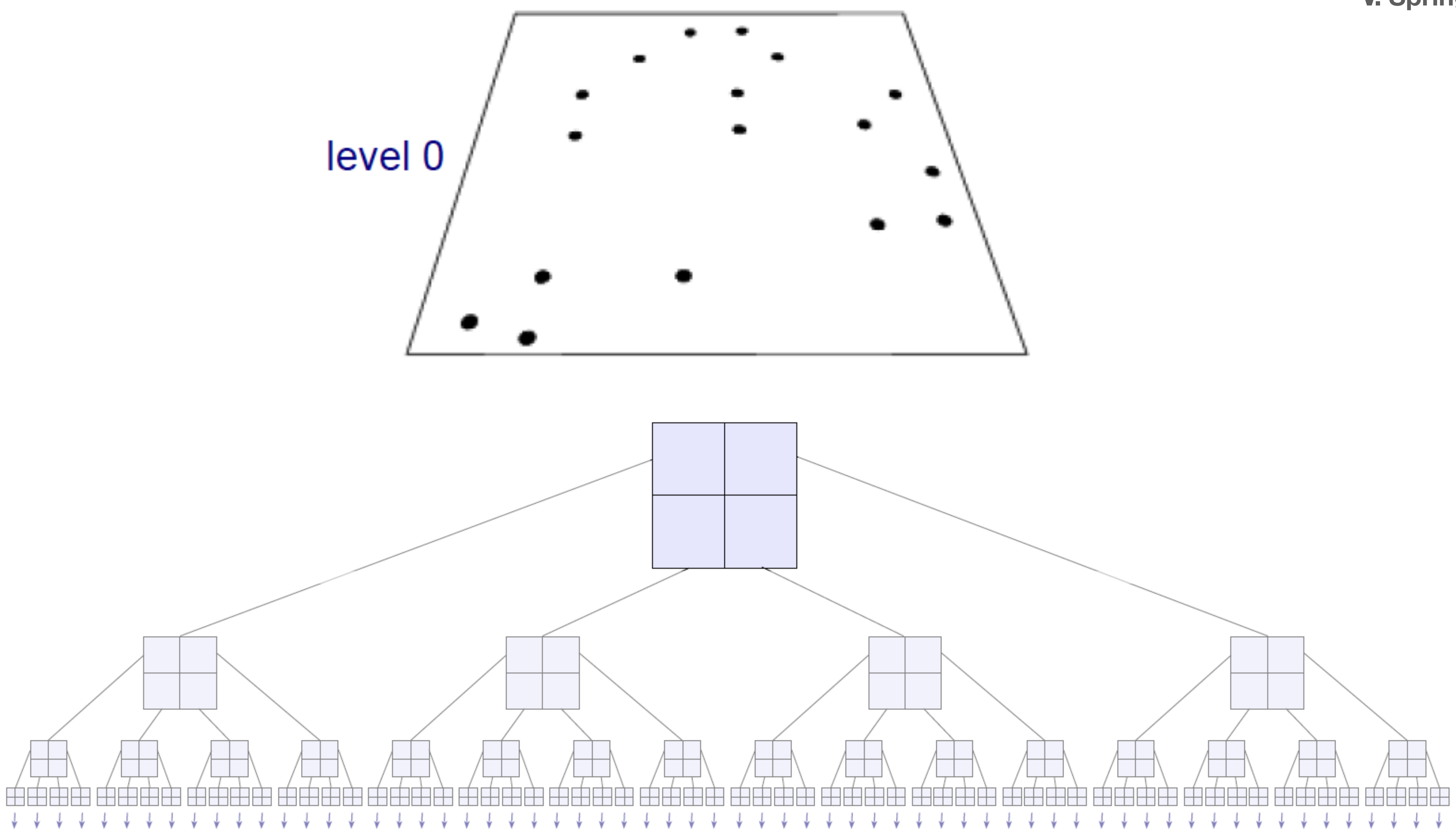
And so on until child nodes have at most 1 particle

CHILD nodes - level 2 (1/64th of the simulation domain)

Solving gravity: the Tree method

This results in a Tree structure of the nodes:

Figure from
V. Springel



Solving gravity: the Tree method

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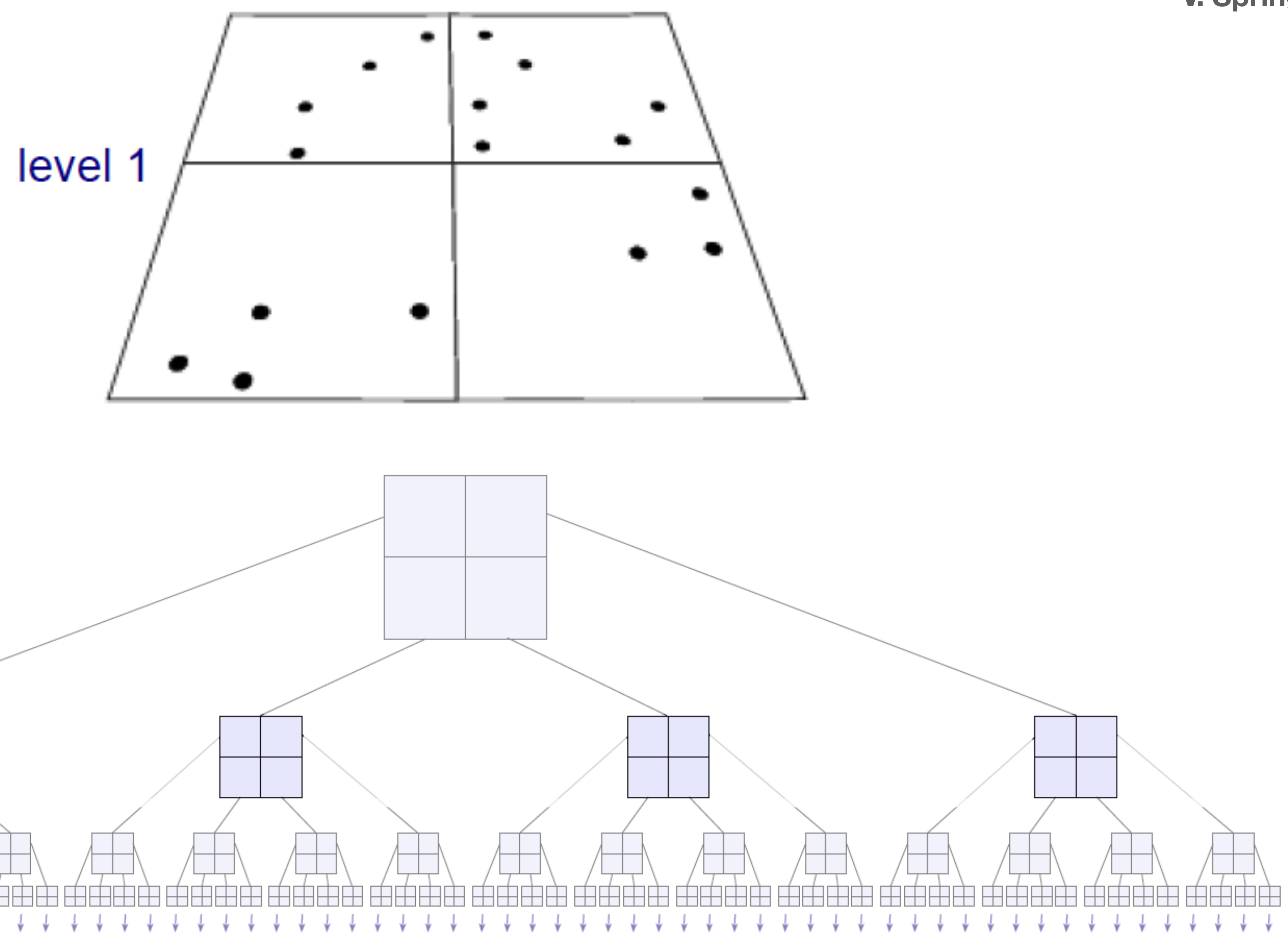

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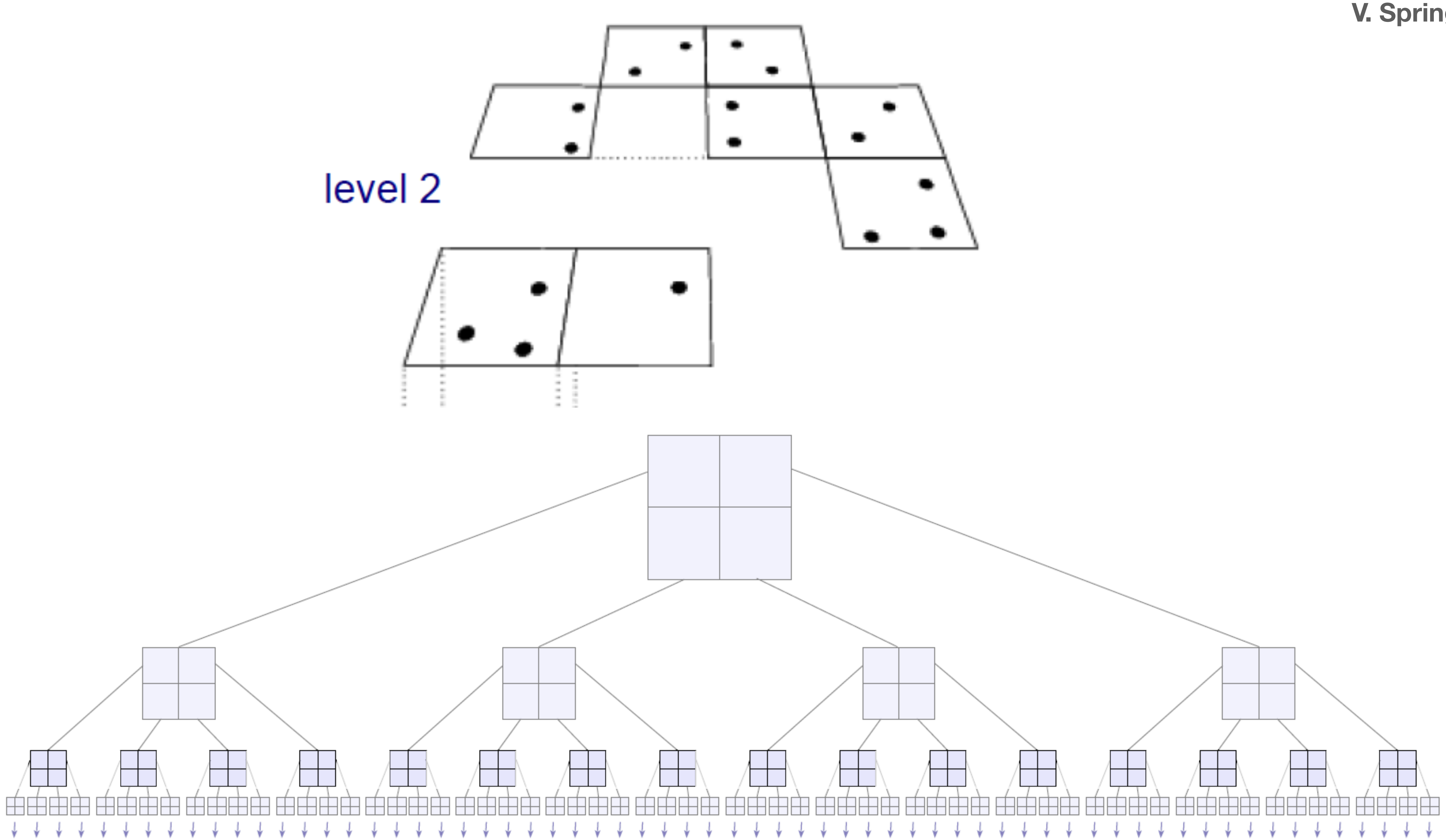

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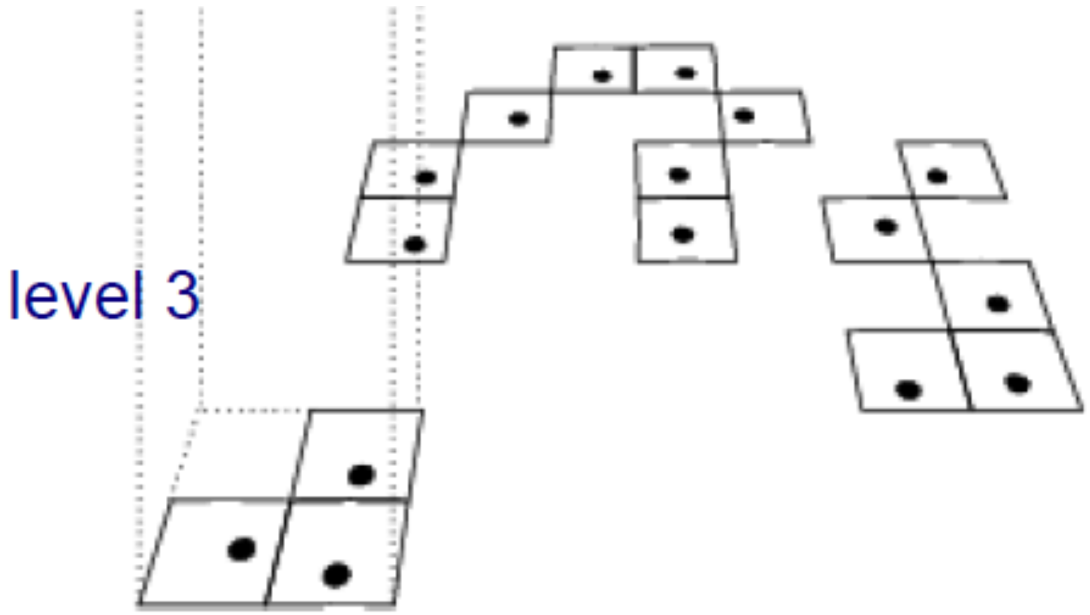

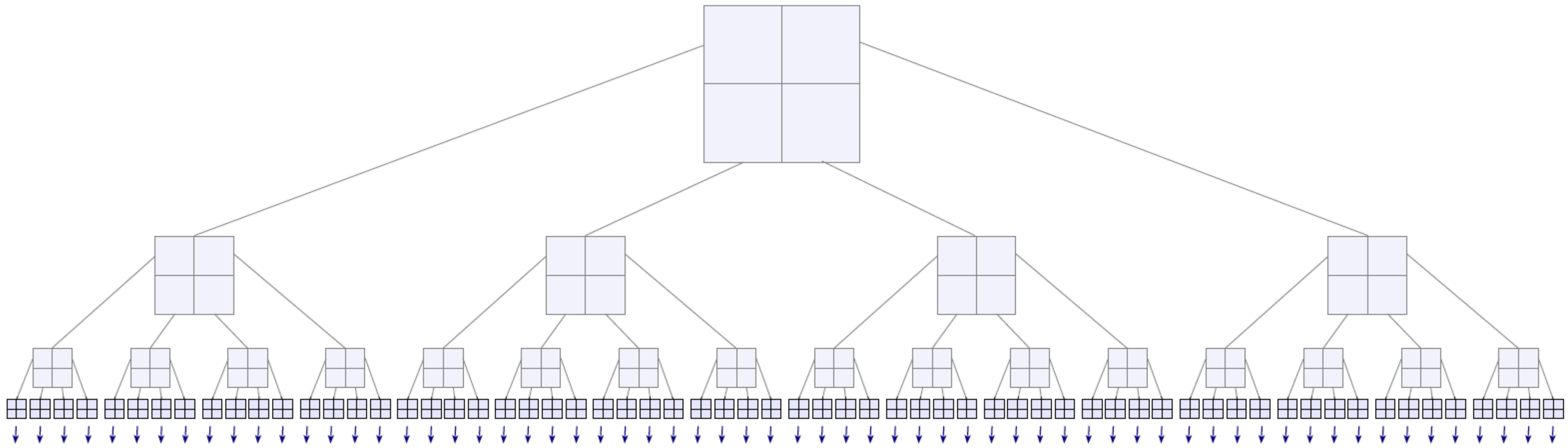

Solving gravity: the Tree method

Why is the tree advantageous?

Solving gravity: the Tree method

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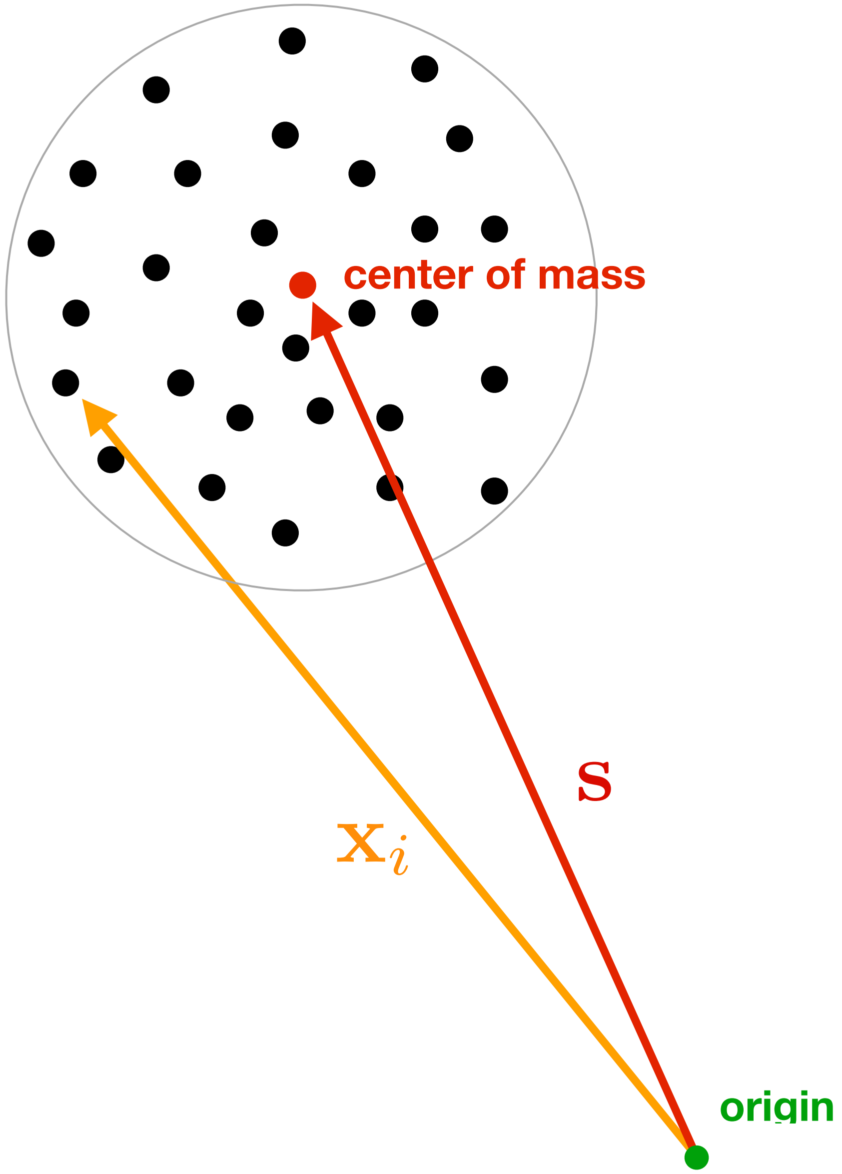
Consider a group of particles at positions x_i with their center of mass at position s.

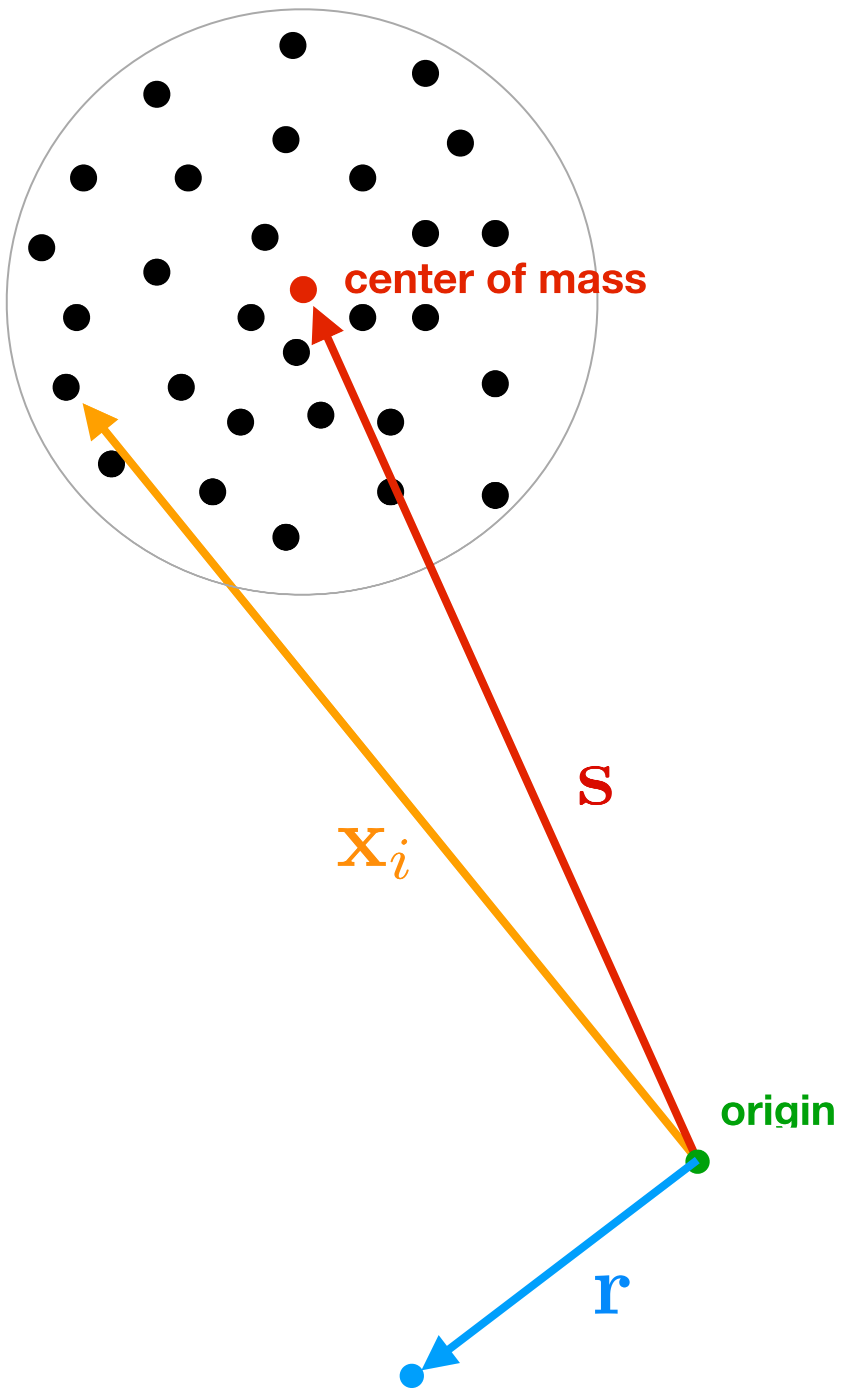


Solving gravity: the Tree method

Why is the tree advantageous?

Consider a group of particles at positions x_i with their center of mass at position s.

We want to compute their resulting gravitational potential on the target particle at position r



Solving gravity: the Tree method

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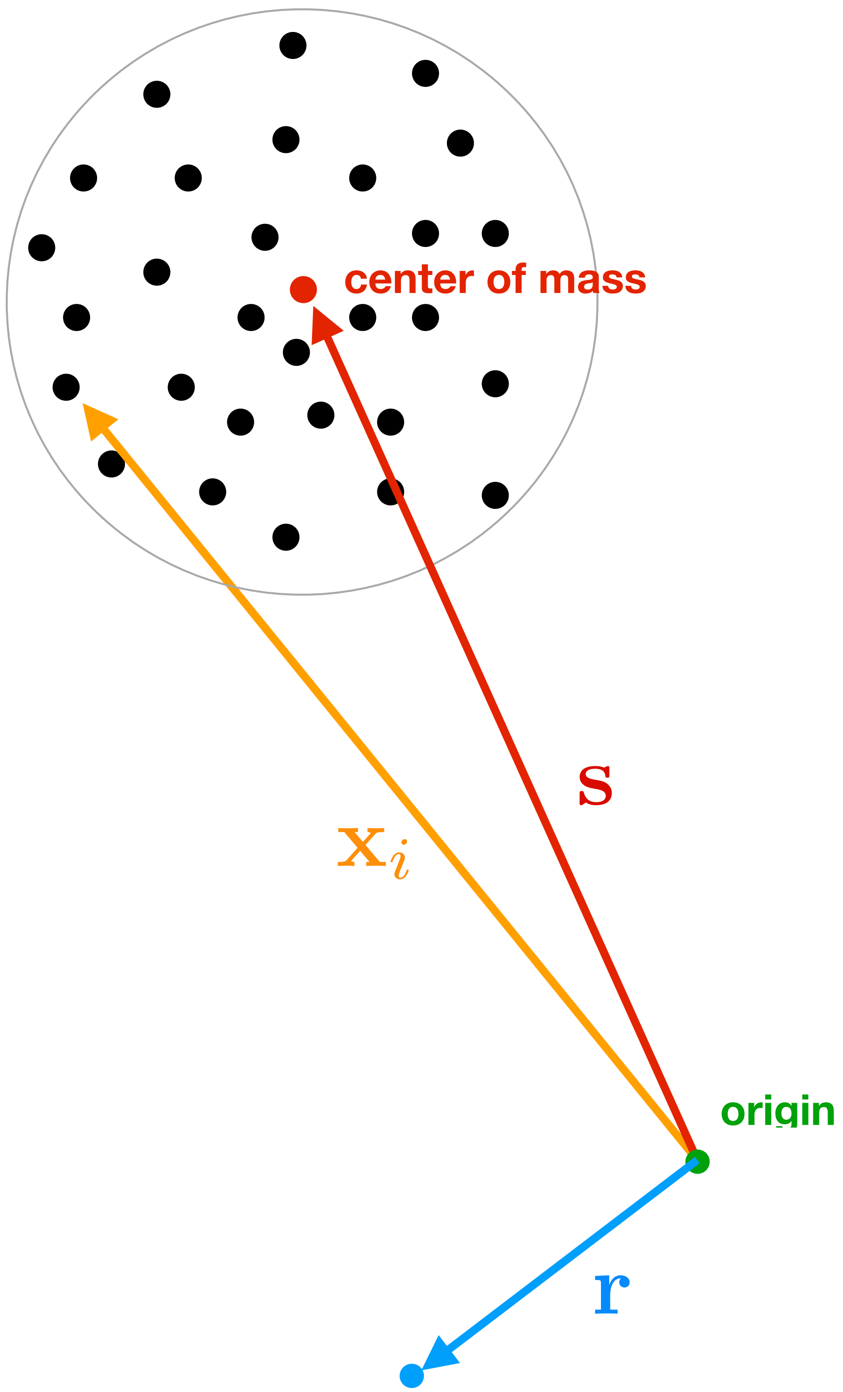
So, we want to compute:

$$\Phi(\mathbf{r}) = -G \sum_i \frac{m_i}{|\mathbf{r} - \mathbf{x}_i|}$$



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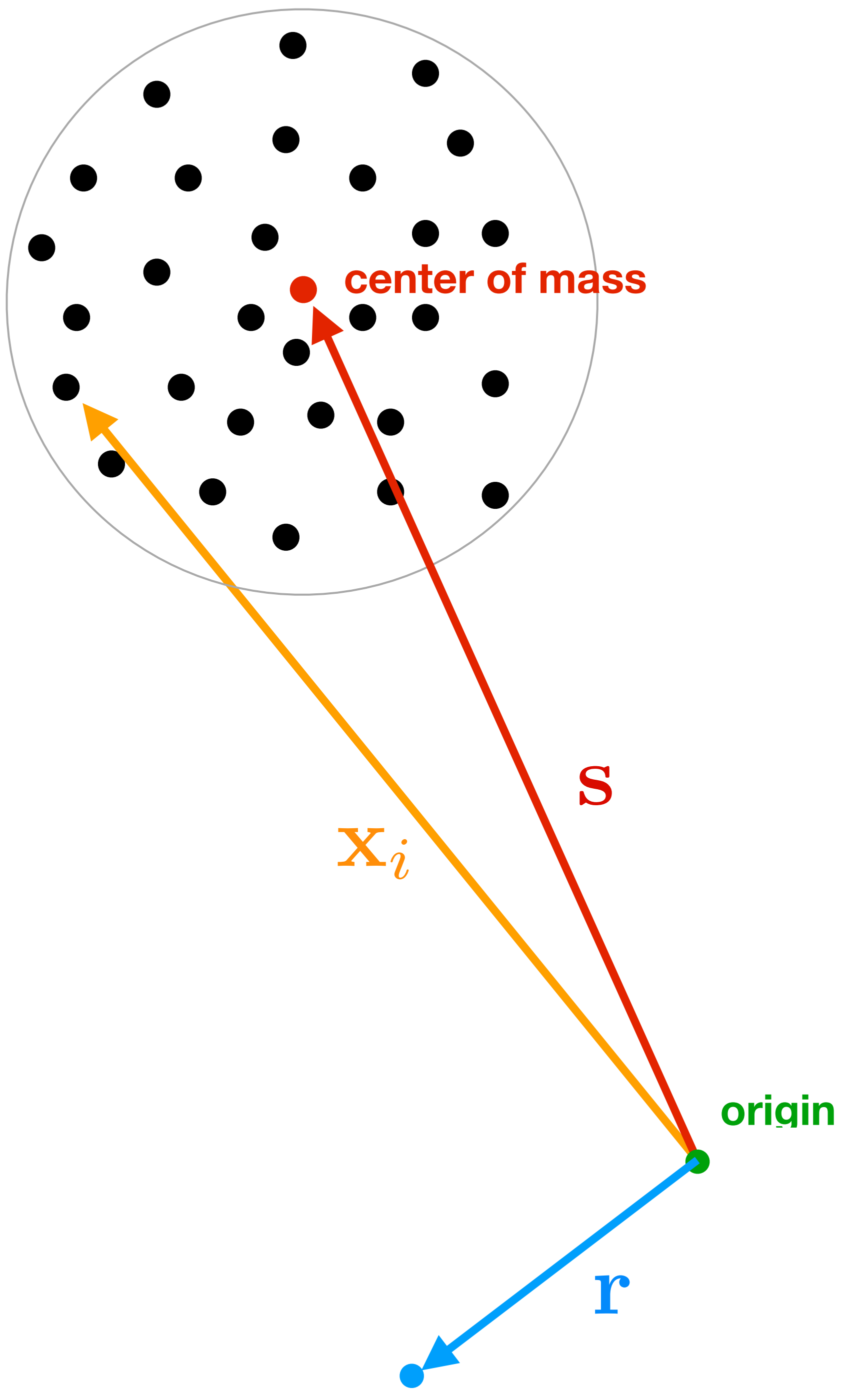


So, we want to compute:

$$\Phi(\mathbf{r}) = -G\sum_i \frac{m_i}{|\mathbf{r}-\mathbf{x}_i|}$$

we can expand the factor:

$$\frac{1}{|\mathbf{r}-\mathbf{x}_i|} = \frac{1}{|(\mathbf{r}-\mathbf{s})-(\mathbf{x}_i-\mathbf{s})|}$$

for

$$|\mathbf{x}_i - \mathbf{s}| \ll |\mathbf{y}| \quad \text{with} \quad \mathbf{y} \equiv |\mathbf{r}-\mathbf{s}|$$

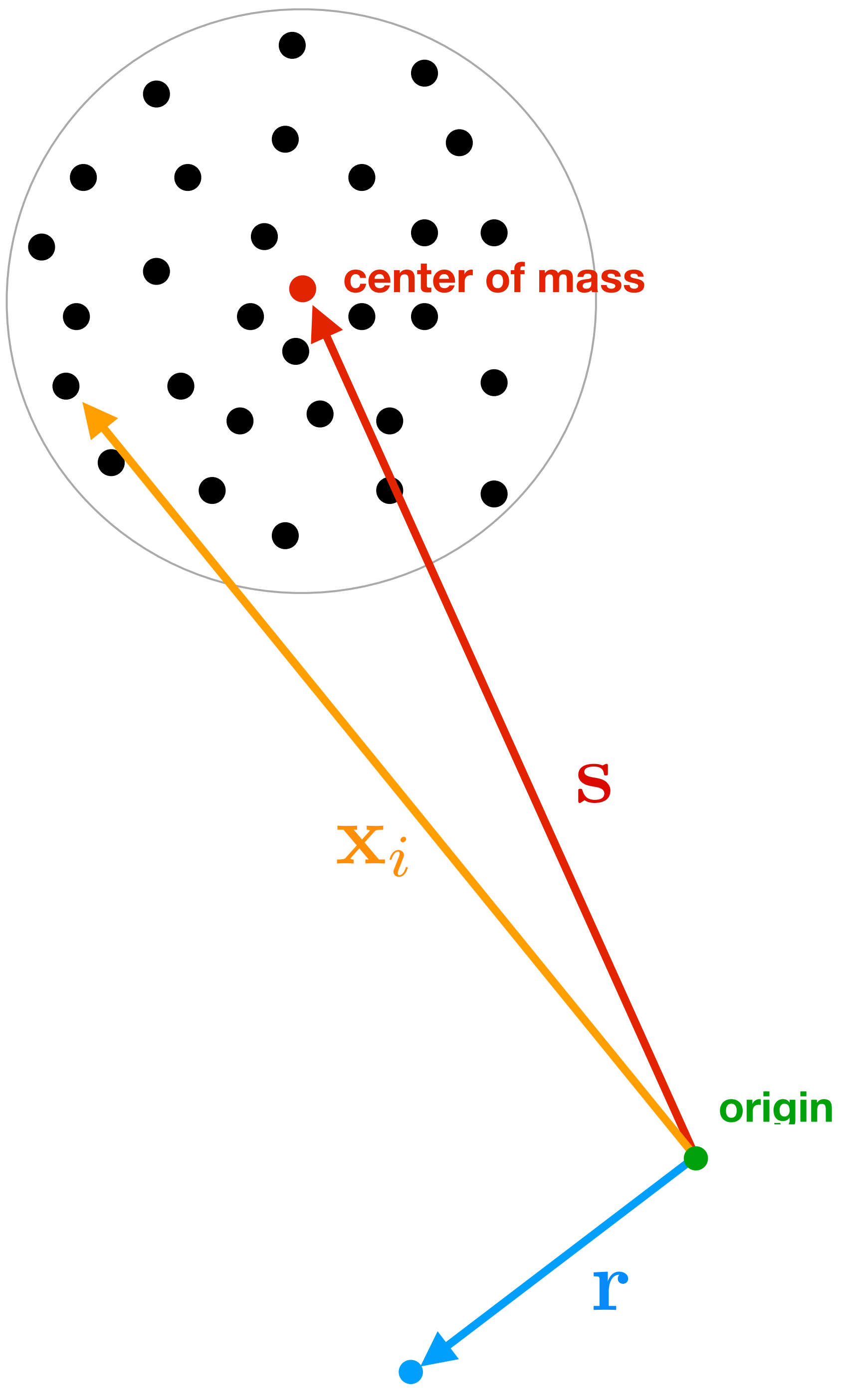
Solving gravity: the Tree method

Why is the tree advantageous?

One gets (with some tedious calculations):

$$\frac{1}{|\mathbf{y}+\mathbf{s}-\mathbf{x}_i|} = \frac{1}{|\mathbf{y}|} - \frac{\mathbf{y}\cdot(\mathbf{s}-\mathbf{x}_i)}{|\mathbf{y}|^3} +$$

$$+\frac{1}{2}\frac{\mathbf{y}^T\left[3(\mathbf{s}-\mathbf{x}_i)(\mathbf{s}-\mathbf{x}_i)^T - \mathbf{I}(\mathbf{s}-\mathbf{x}_i)^2\right]\mathbf{y}}{|\mathbf{y}|^5} + \ldots$$



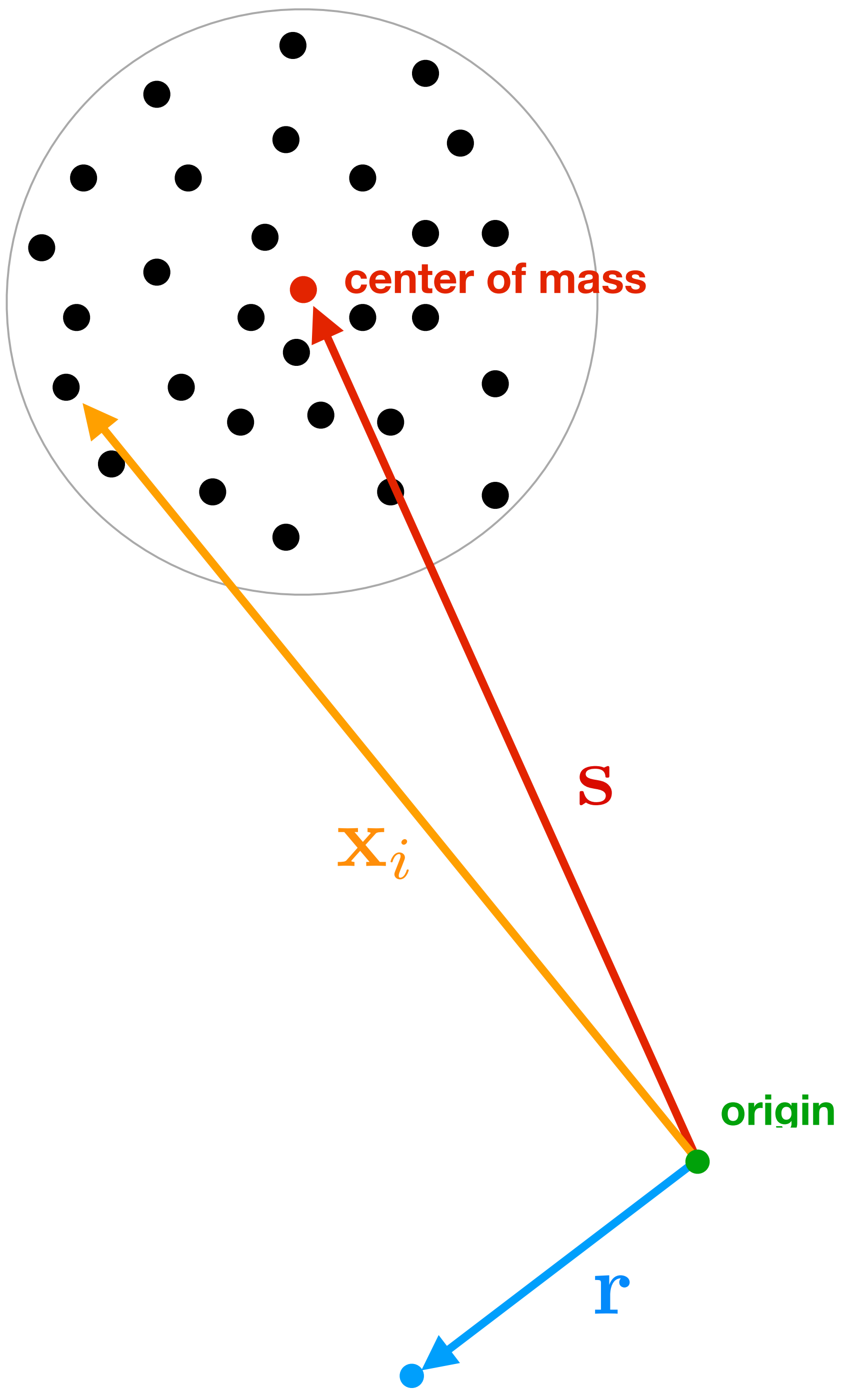
Solving gravity: the Tree method

Why is the tree advantageous?

One gets (with some tedious calculations):

vanishes when summed over all nodes particles

$$\frac{1}{|\mathbf{y}+\mathbf{s}-\mathbf{x}_i|} = \frac{1}{|\mathbf{y}|} - \frac{\mathbf{y}\cdot(\mathbf{s}-\mathbf{x}_i)}{|\mathbf{y}|^3} + \frac{1}{2}\frac{\mathbf{y}^T\left[3(\mathbf{s}-\mathbf{x}_i)(\mathbf{s}-\mathbf{x}_i)^T - \mathbf{I}(\mathbf{s}-\mathbf{x}_i)^2\right]\mathbf{y}}{|\mathbf{y}|^5} + \ldots$$



Solving gravity: the Tree method

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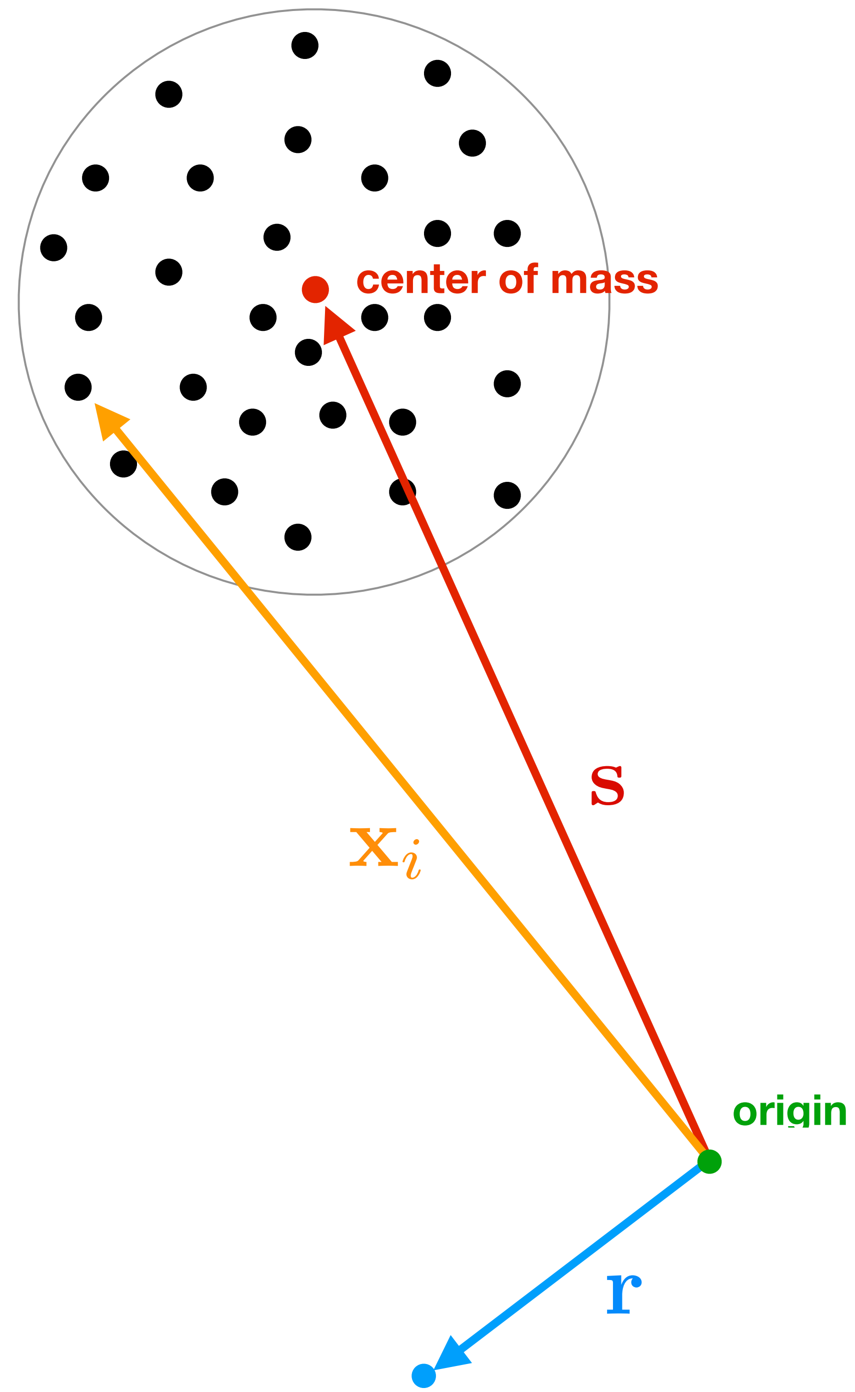
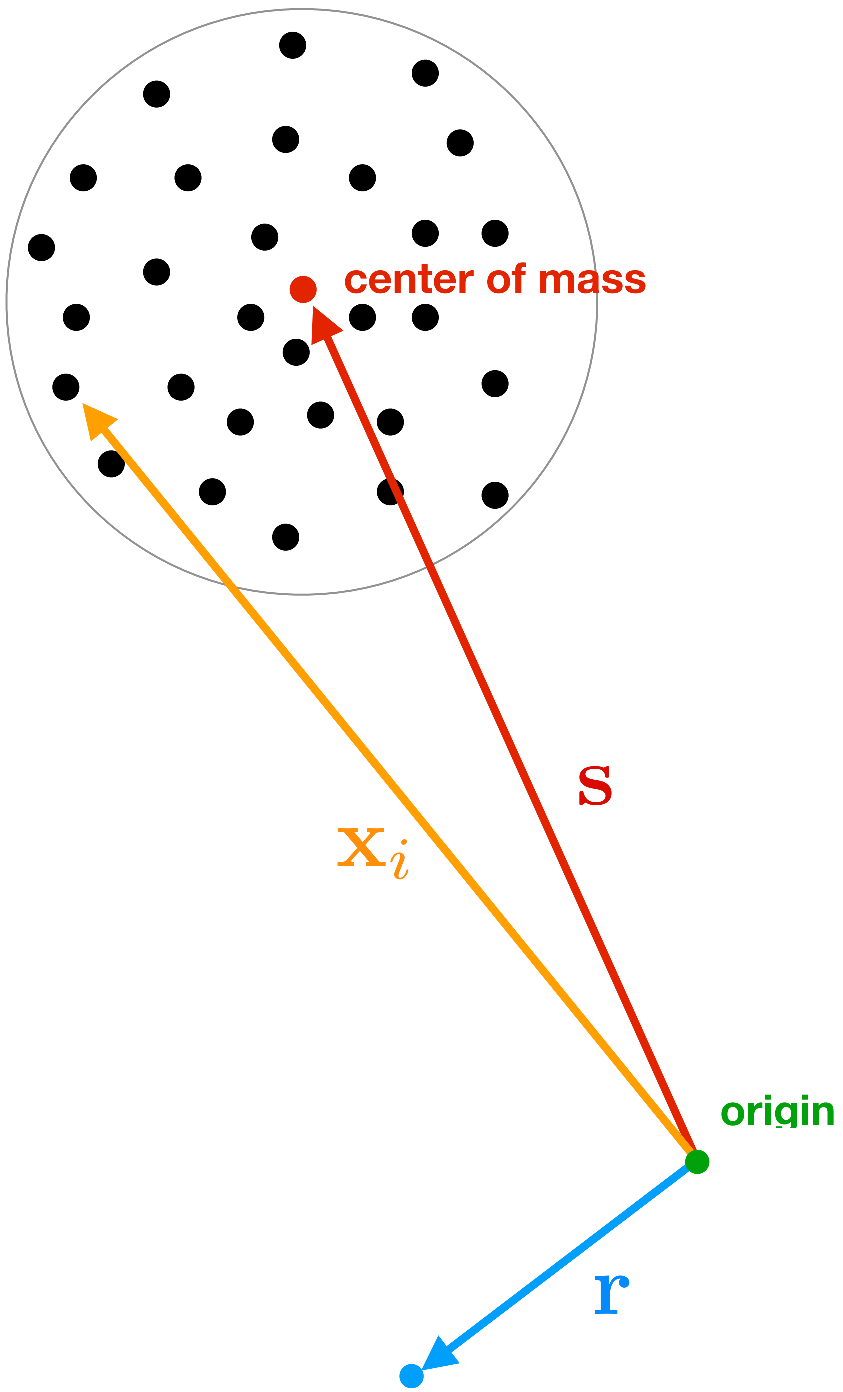


Solving gravity: the Tree method

Why is the tree advantageous?

The group of particles can correspond to a particular Tree node

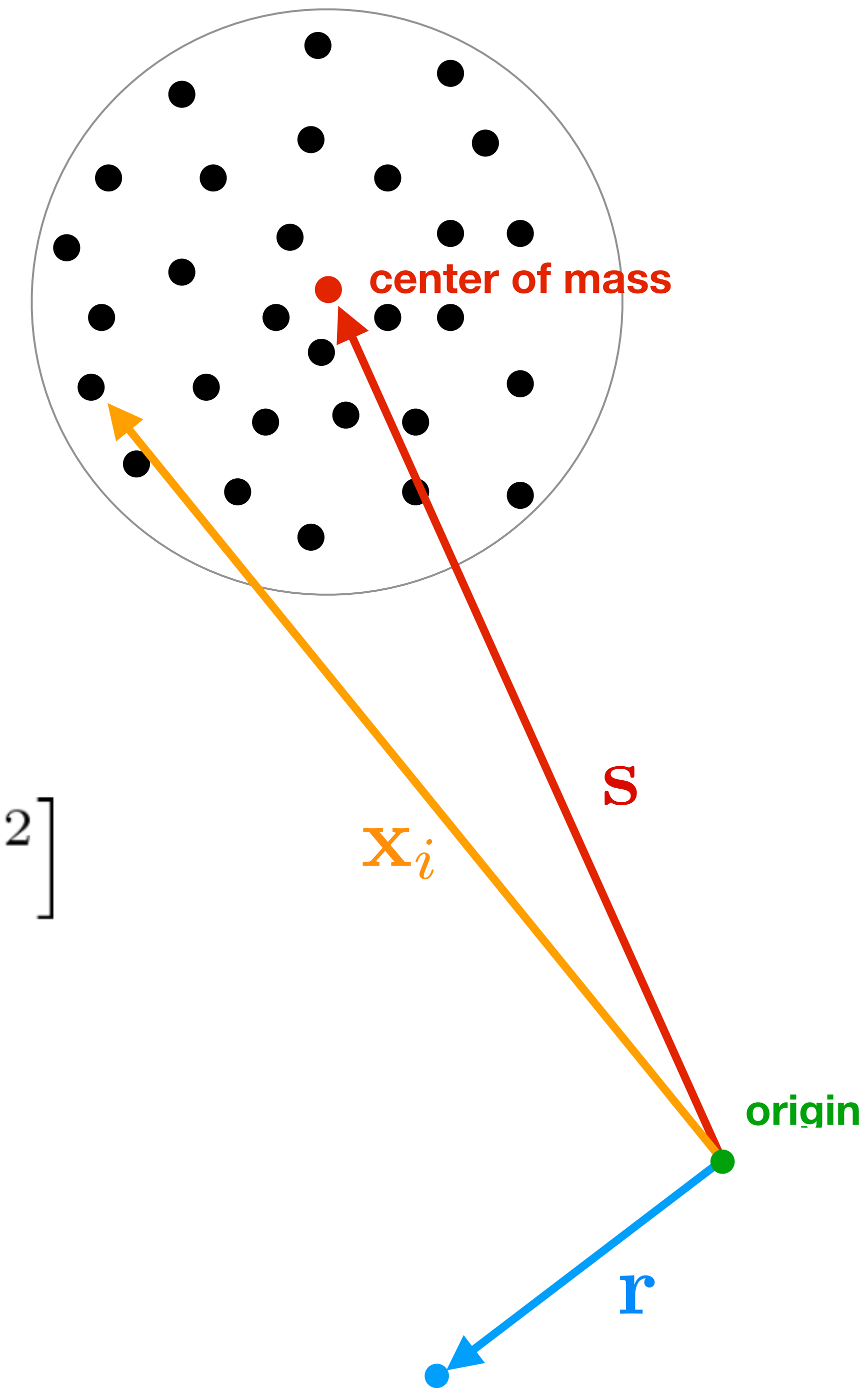


Solving gravity: the Tree method

Why is the tree advantageous?

The group of particles can correspond to a particular Tree node

So the following multipole moments can be computed and stored for all tree nodes:



- Monopole:

$$M = \sum_i m_i$$

- Quadrupole tensor:

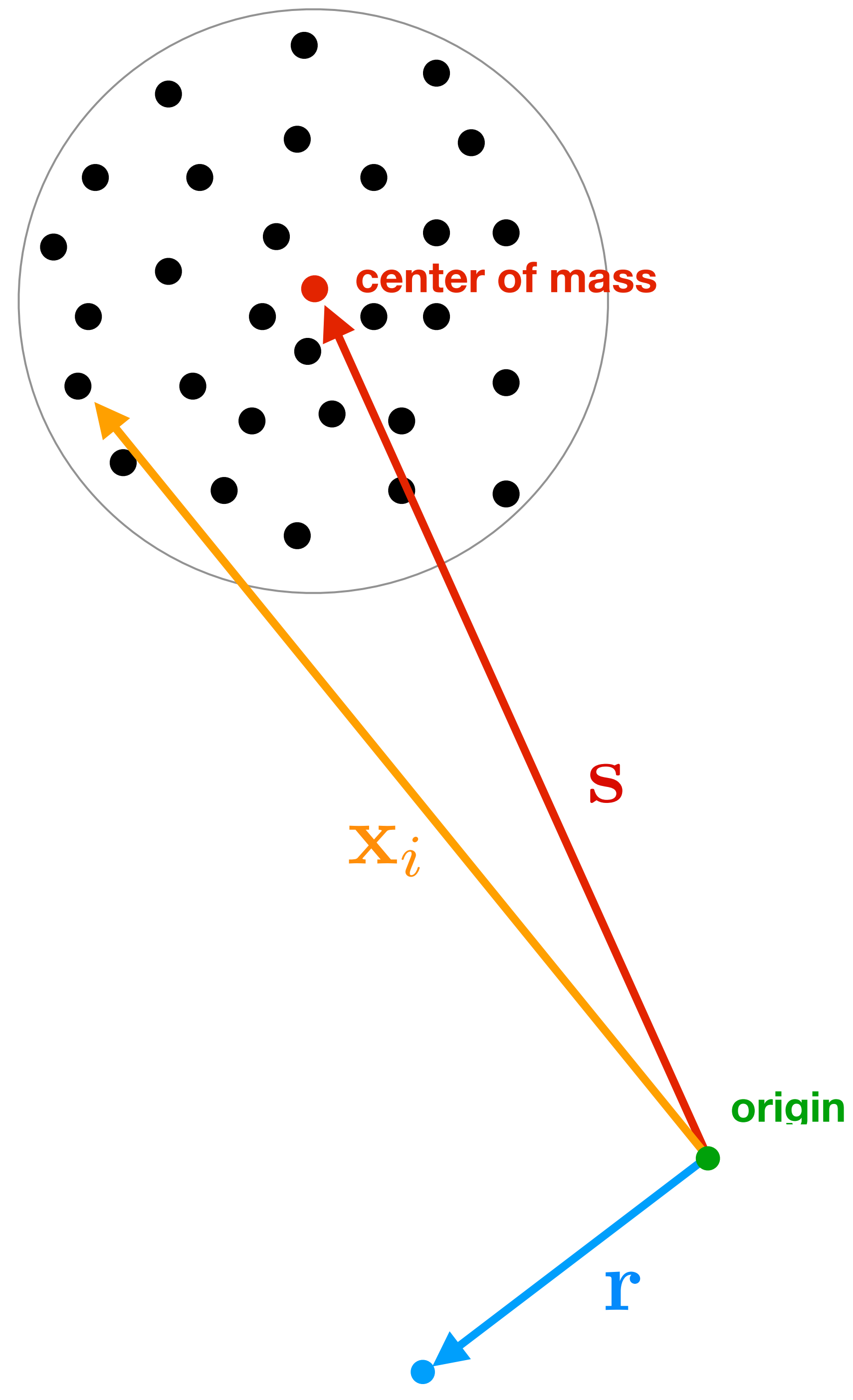
$$Q_{ij} = \sum_k m_k \left[3(\mathbf{x}_k - \mathbf{s})_i(\mathbf{x}_k - \mathbf{s})_j - \delta_{ij}(\mathbf{x}_k - \mathbf{s})^2\right]$$

and the resulting potential is:

$$\Phi(\mathbf{r}) = -G\left[\frac{M}{|\mathbf{y}|} + \frac{1}{2}\frac{\mathbf{y}^T\mathbf{Q}\,\mathbf{y}}{|\mathbf{y}|^5}\right]$$

Solving gravity: the Tree method

The Tree algorithm is approximating the collective contribution to the gravitational potential of groups of “distant” particles with its multipole expansion.



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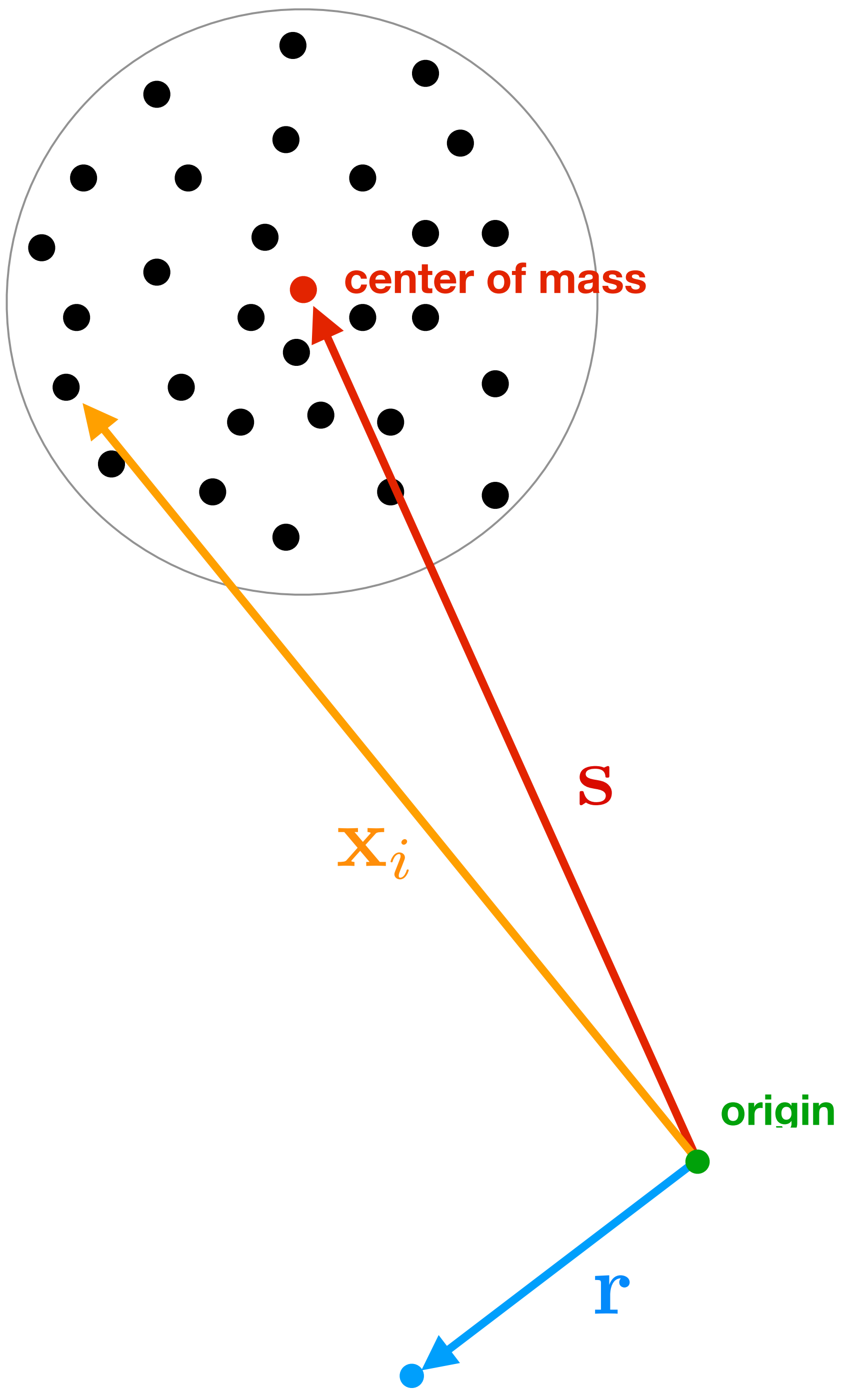
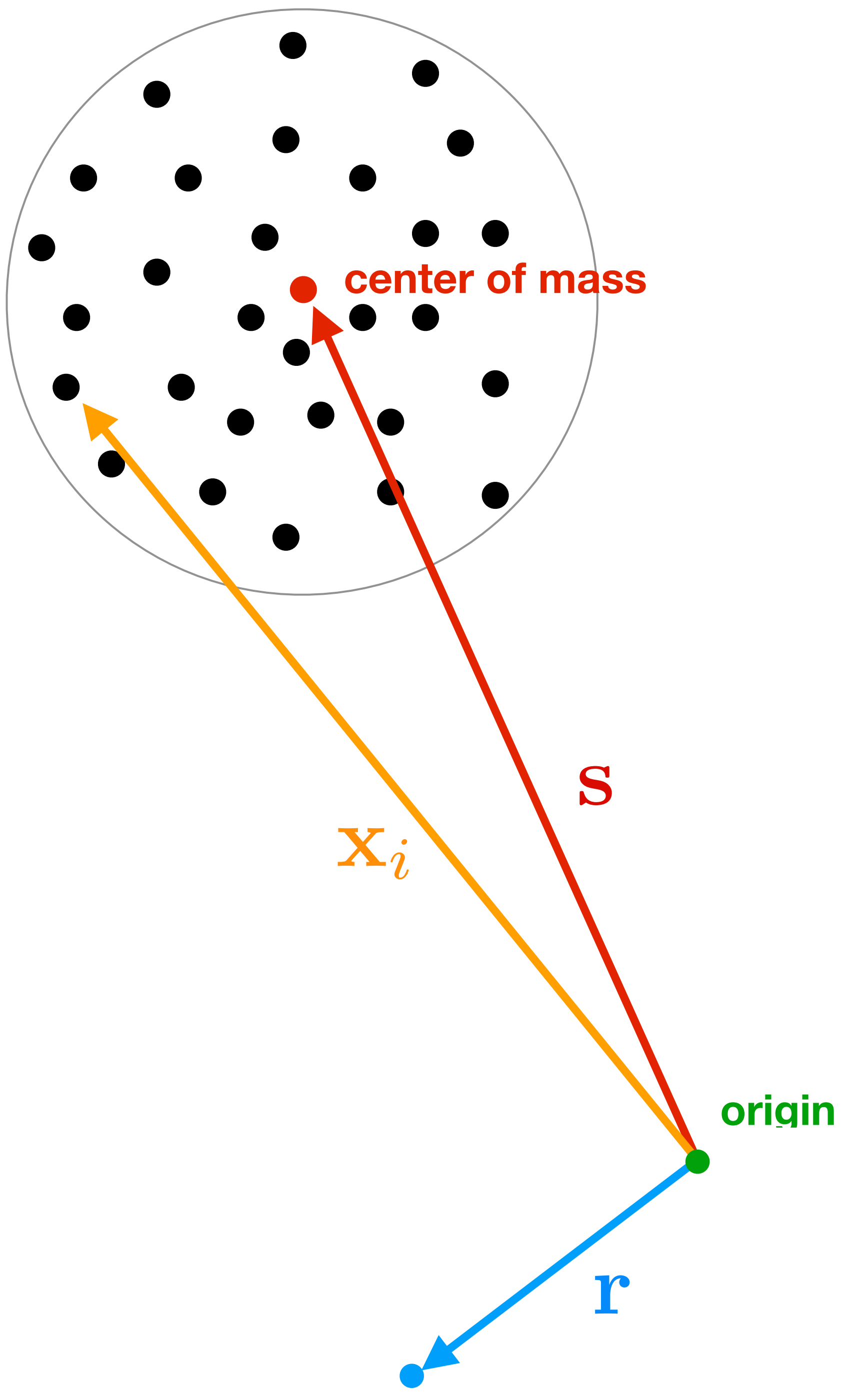


Often (as e.g. in Gadget2-3) only the Monopole is used, so that one can replace the N_{node} interactions with a single interaction for its center of mass:

$$-G\sum_{i=1}^{N_{\rm node}}\frac{m_i}{|\mathbf{r}-\mathbf{x}_i|} \rightarrow -G\frac{M}{|\mathbf{y}|}$$

Solving gravity: the Tree method

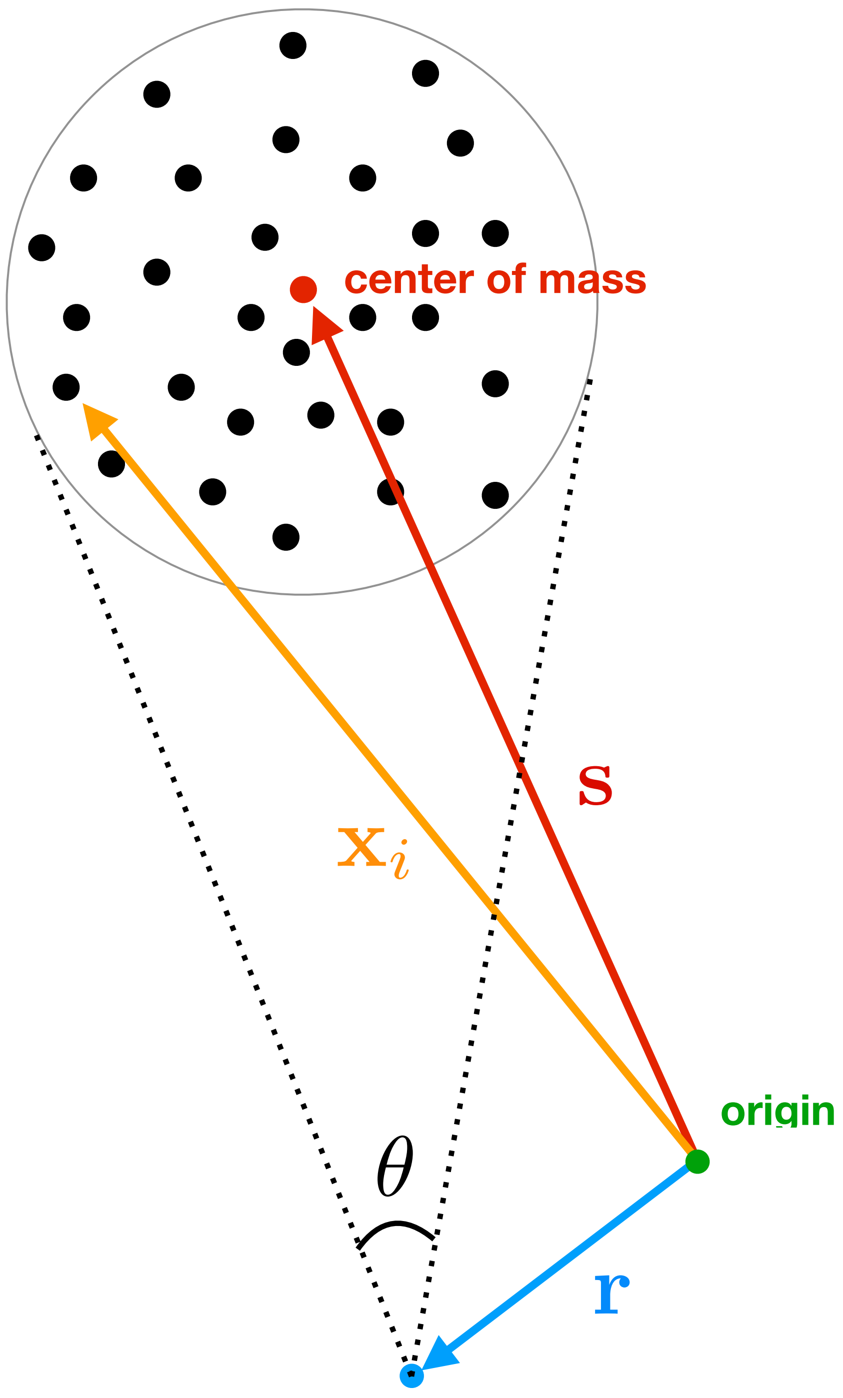
The **accuracy of the approximation can be adjusted by defining what "distant" means**.



Solving gravity: the Tree method

The **accuracy of the approximation can be adjusted by defining what "distant" means**.

This is done by setting **a threshold for the angle** θ under which a given tree node (i.e. a group of particles) is "seen" from the target particle on which the potential is to be computed. Such threshold (dividing sufficiently distant nodes from closer ones) is called **OPENING ANGLE**



Solving gravity: the Tree method

Advantages and disadvantages of the Tree method

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- There is **no intrinsic restriction to the dynamic range** that can be achieved: spatial resolution automatically increases in regions where particles cluster
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The main **disadvantage** of the tree method is that for highly homogeneous matter distributions (as e.g. the cosmic density field at high redshifts) **the almost vanishing force on each particle is the result of the cancellation of many larger contributions**. This makes it numerically expensive to obtain high accuracy in the force calculation.

How?

Solving gravity: the Tree-PM method

(if you can't beat them, join'em!)

Solving gravity: the TreePM method

In order to overcome the latter disadvantage of the Tree method, and to exploit the high speed of the PM method, many N-body codes implement a combination of the Tree and PM algorithms, called the Tree-PM method. The idea is the following:

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$$\phi_k^{\mathrm{short}} = \phi_k[1 - \exp(-k^2 r_s^2)]$$

4: Inverse-Fourier transform this to real space and solve with Tree

$$\phi^{\mathrm{short}}(r) = \mathcal{F}^{-1}(\phi_k^{\mathrm{short}}) = -\frac{Gm}{r}\mathrm{erfc}\left(\frac{r}{2r_s}\right)$$

Solving gravity: the TreePM method

where the short-range potential features the complementary error function

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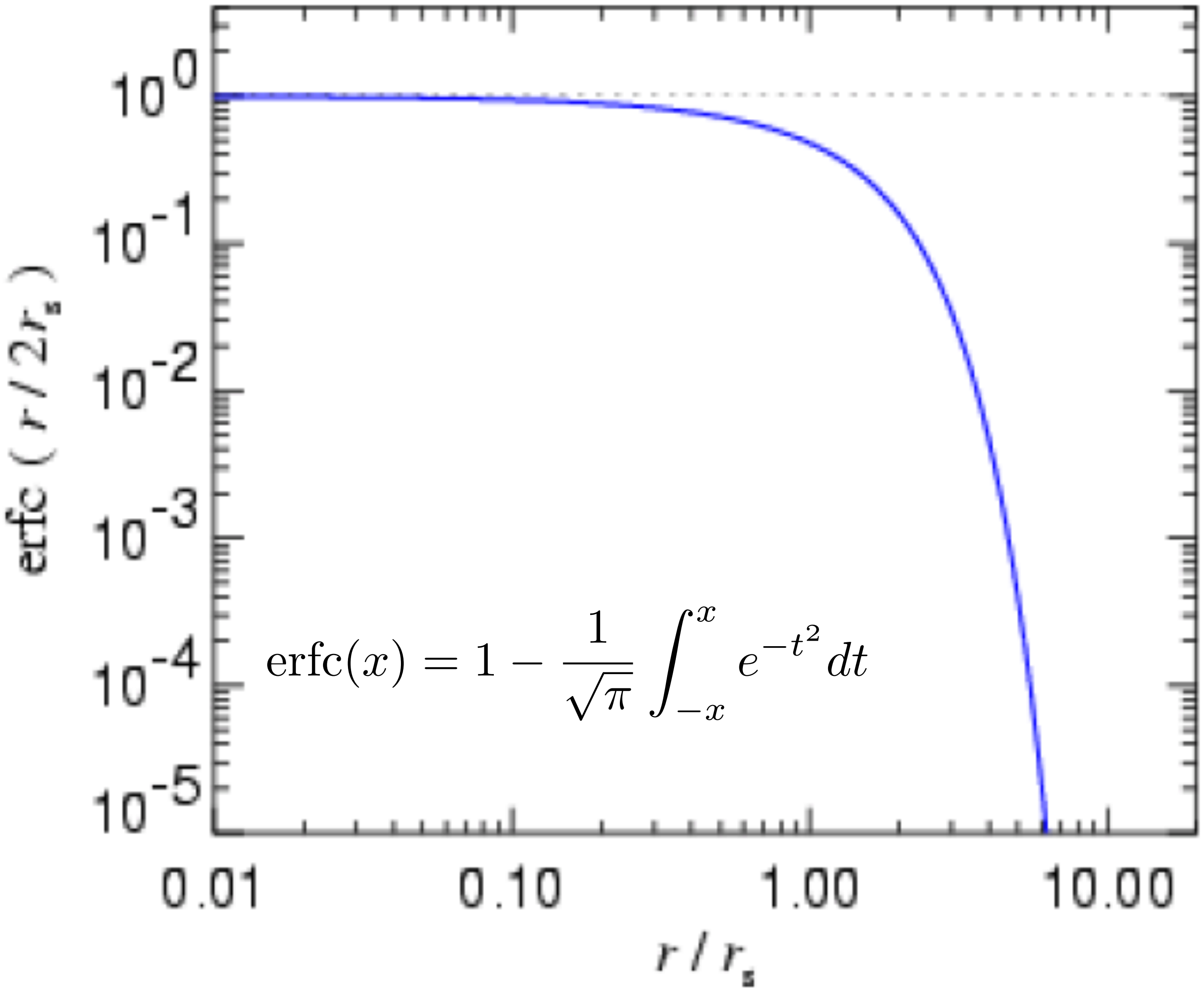

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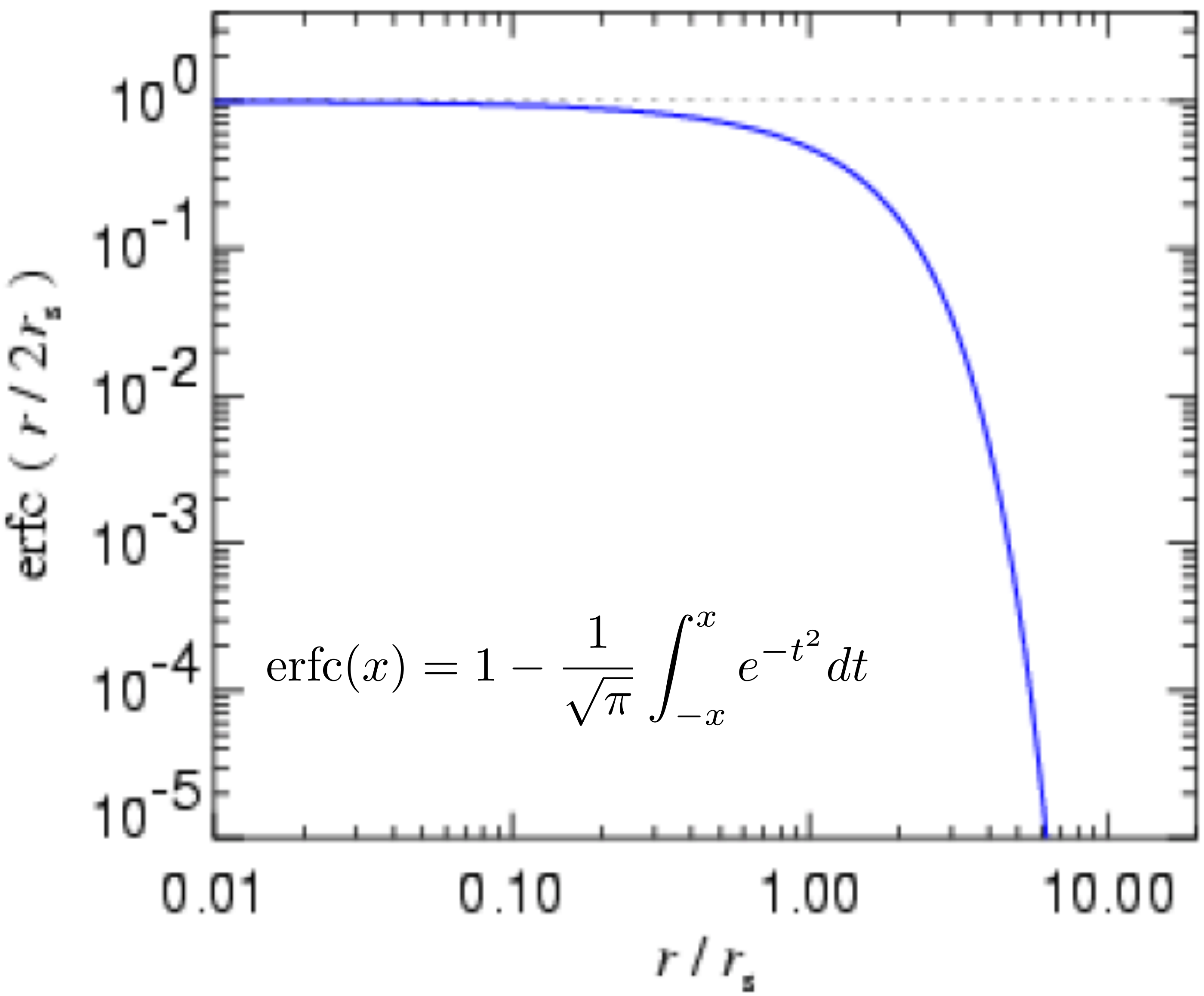
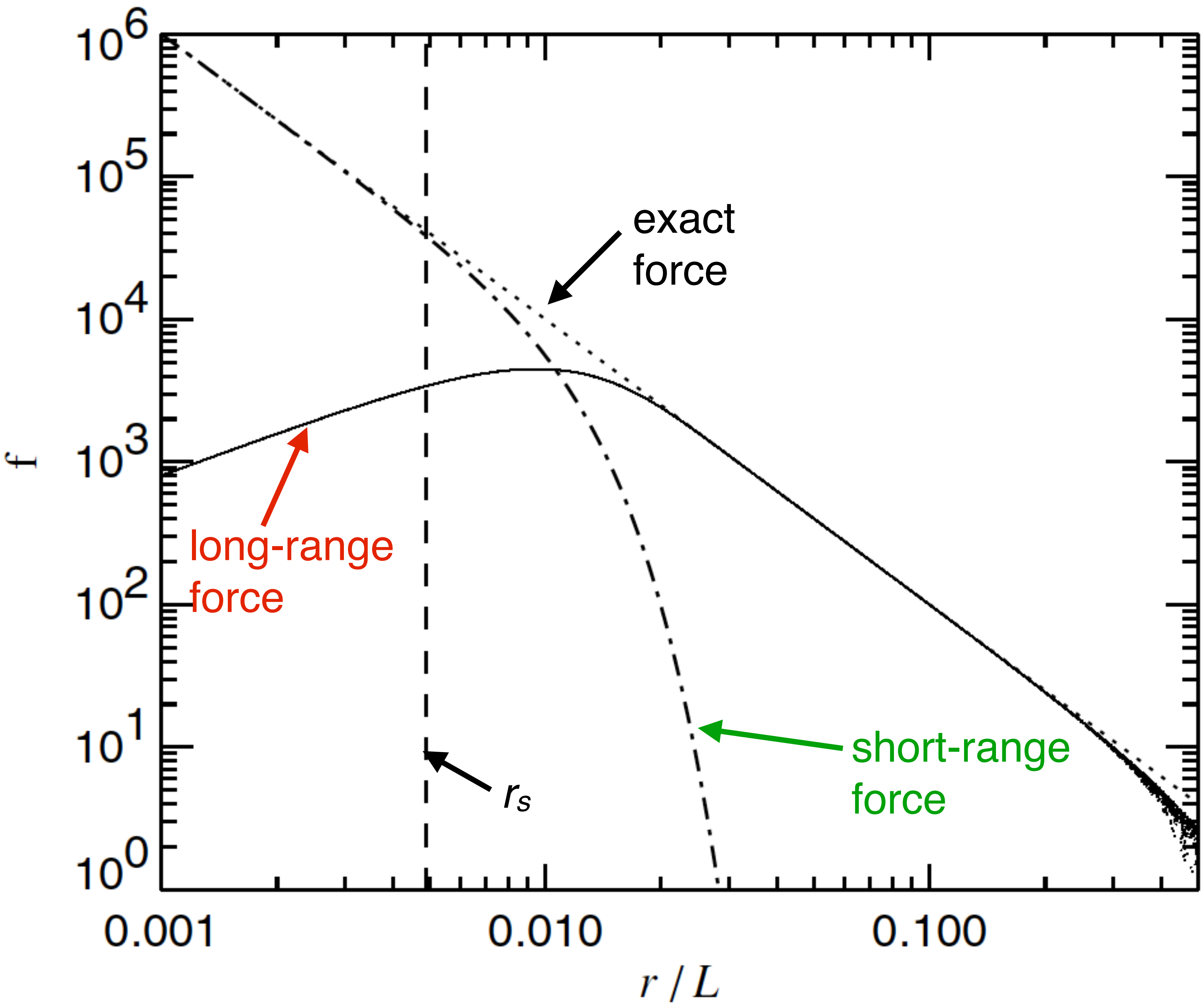


The Tree-PM method retains all the pros of the Tree algorithm, but has **the main advantage of ensuring a fast and accurate computation of long-range forces in all situations.**

Solving gravity: the TreePM method

The Tree-PM force splitting



Plot from Gadget2 code paper
Springel (2005)

Solving gravity: the TreePM method

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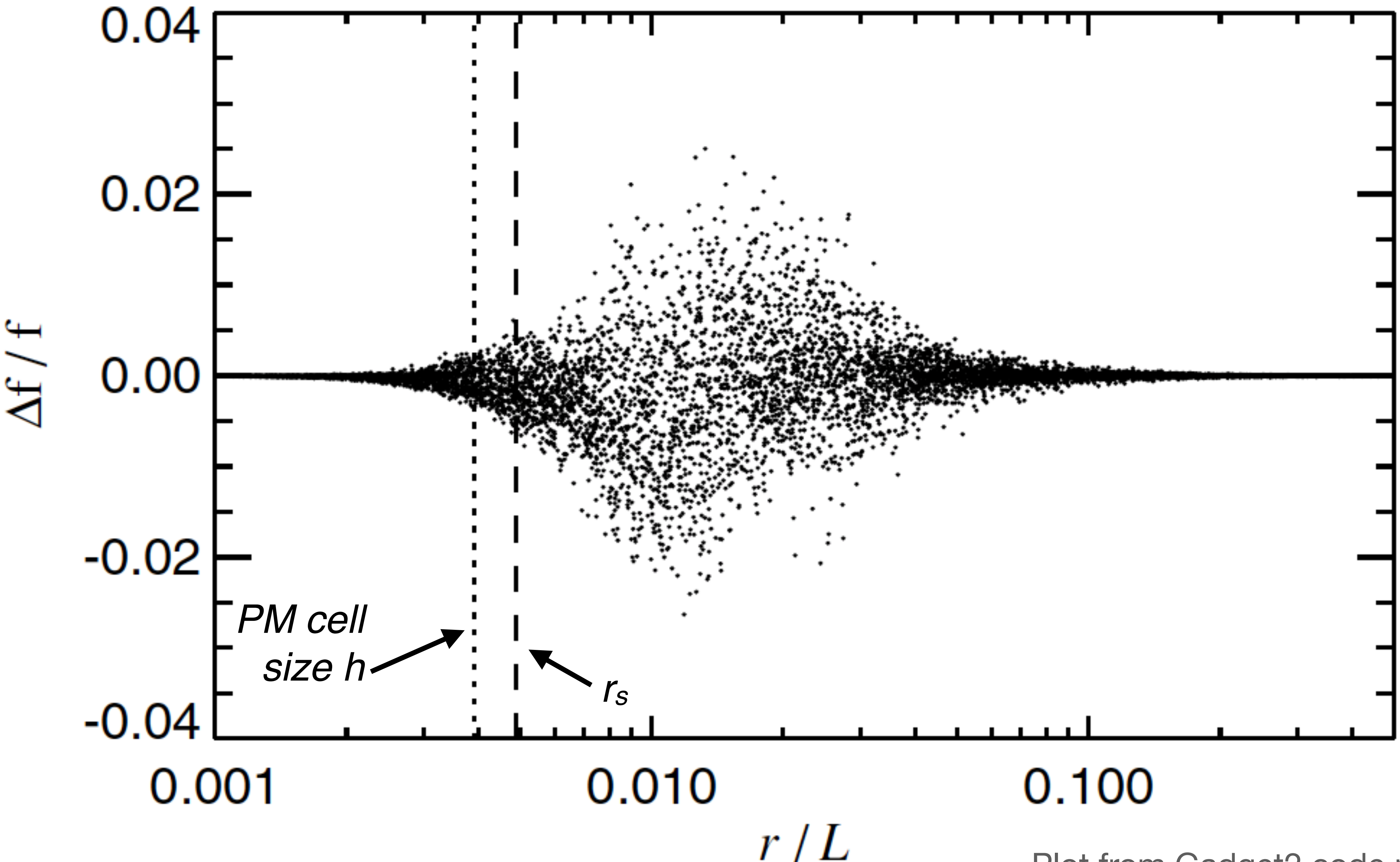

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How?

Solving gravity: Multi-grid methods

Solving gravity: Multigrid Iterative Solvers

A different procedure to solve the Poisson equation is given by approximating the Laplace operator ∇^2 on a grid.

Consider a 1D version of Poisson's equation:

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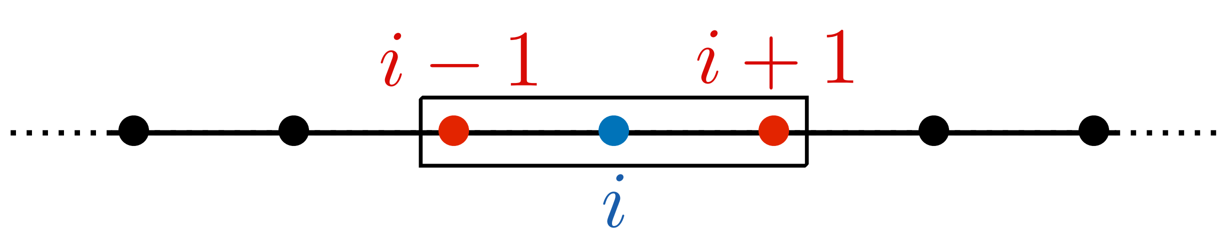
3-points stencil



Solving gravity: Multigrid Iterative Solvers

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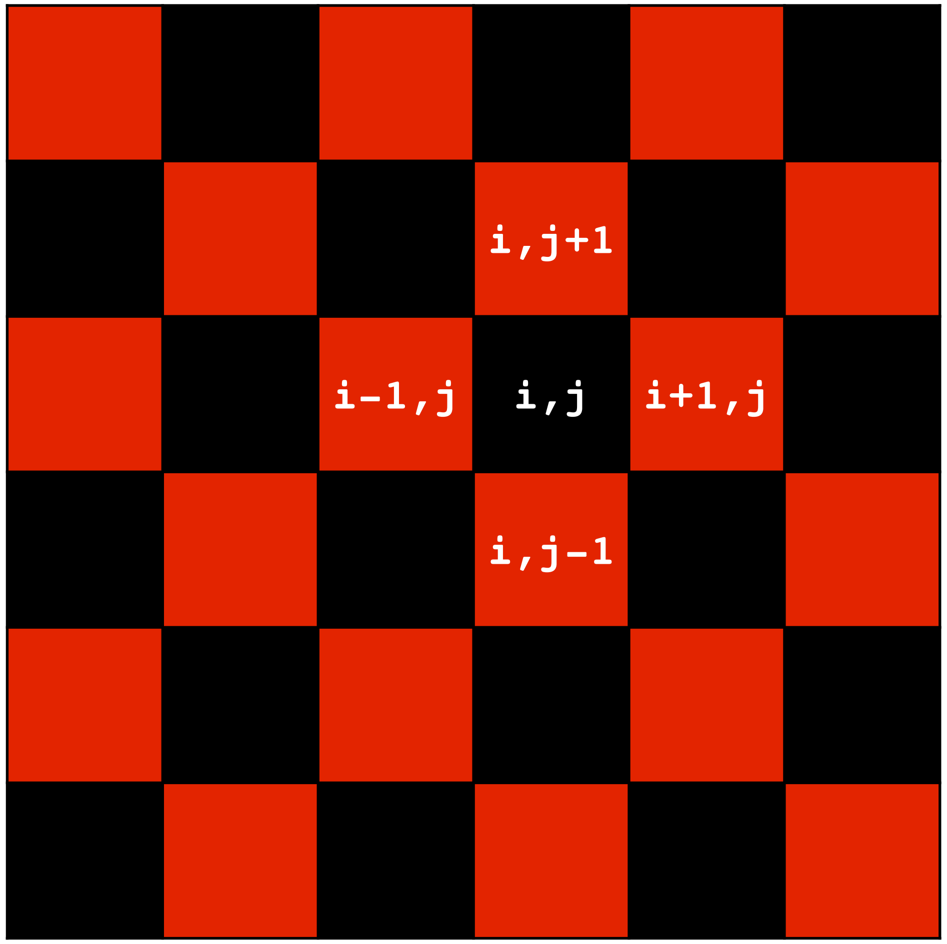

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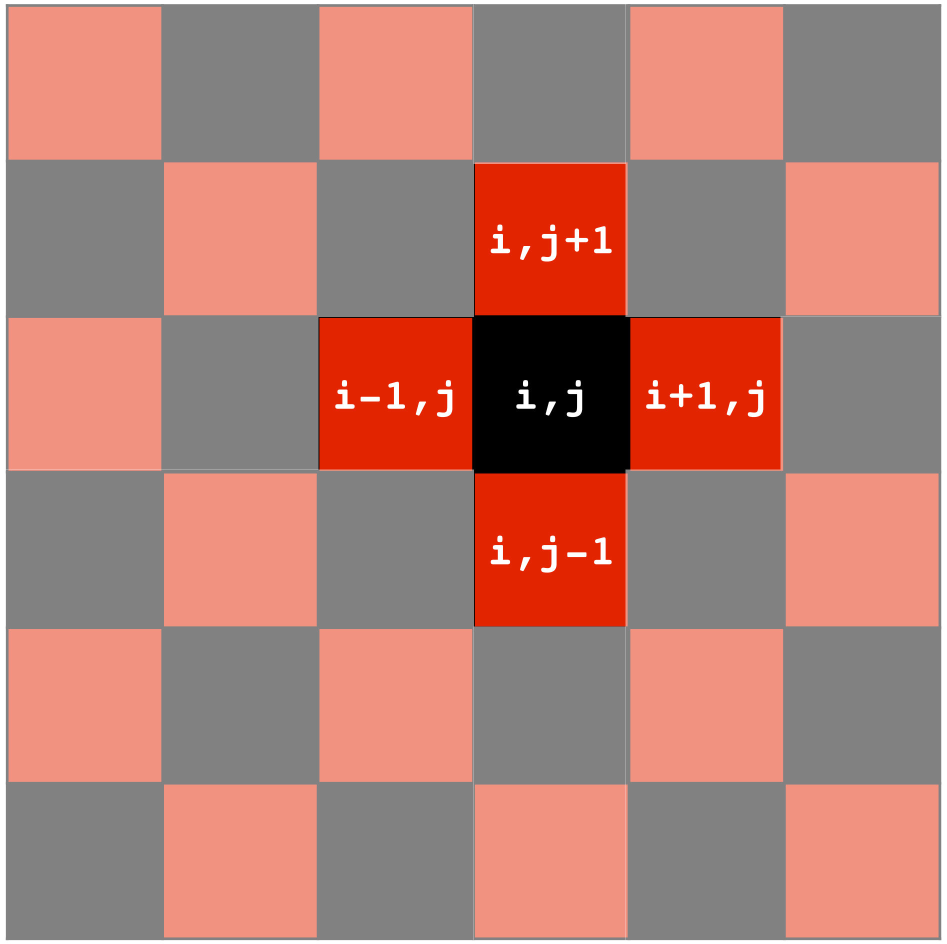

5-points stencil

Update of a black cell depends only on values of red cells: Red-Black sweep: first update half of the cells (all the black ones) and then the other half (all the red ones) using the updated values of the black cells

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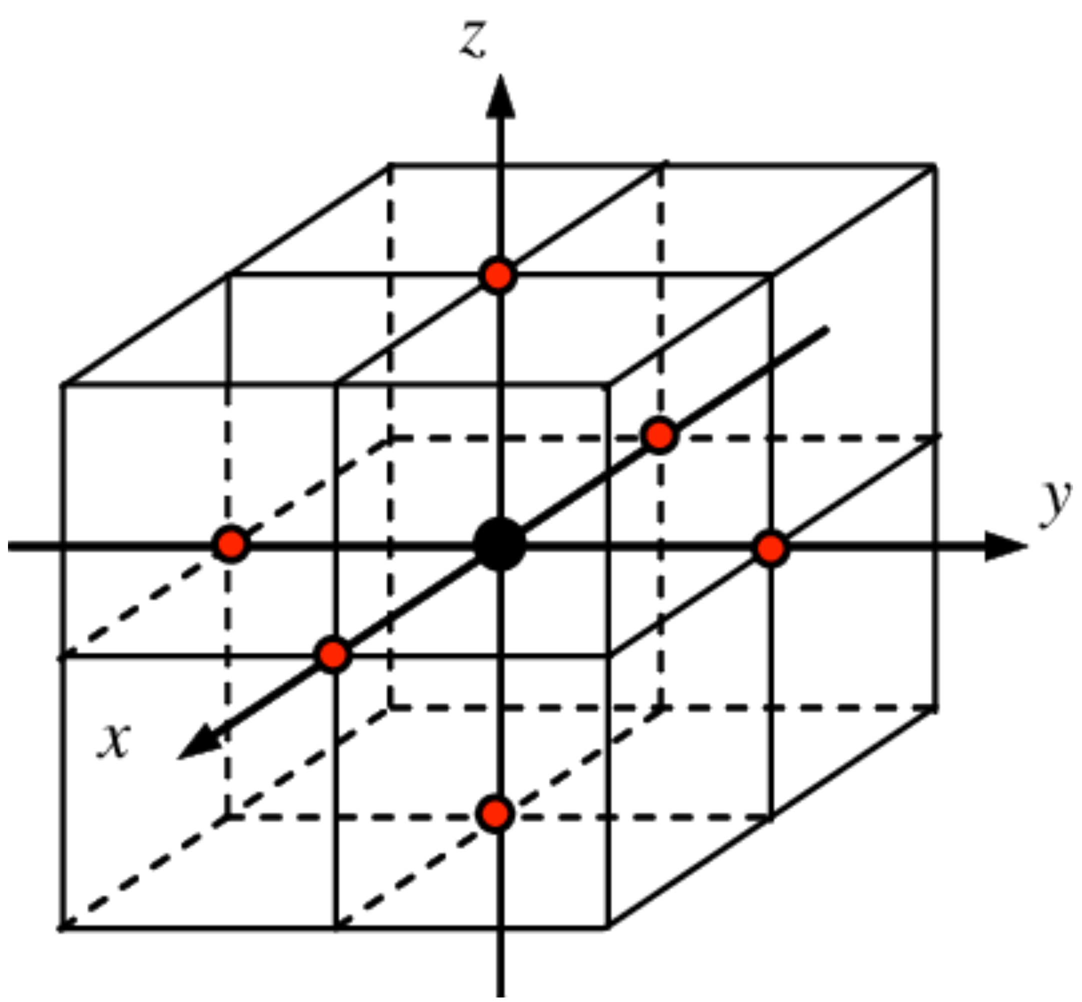

7-points stencil

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Therefore, the solution of Poisson equation corresponds to finding the zeros of the function

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Solving this by matrix inversion is computationally prohibitive for large grids as complexity grows like $\mathcal{O}(N^3)$ but can be solved iteratively using Newton-Raphson's method.

Solving gravity: Multigrid Iterative Solvers

A short recap of Newton-Raphson's method.

For a 1D function $f(x)$ the zeros of the function (i.e. the values $\tilde{x}$ for which $f(\tilde{x}) = 0$) can be found iteratively starting from a guess $\tilde{x}^{(0)}$ with the rule :

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which can be iterated until

$$|\phi_{i,j,k}^{(n+1)} - \phi_{i,j,k}^{(n)}| < \epsilon$$

Solving gravity: Multigrid Iterative Solvers

As only neighbouring points in the grid communicate with each other at every iteration (7-points stencil) information propagates only by one cell per iteration: convergence is slow.

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Convergence can be fastened by using **multi-grid acceleration**:

1) map the target grid guess on a coarser grid (restriction)
2) get a (faster) solution for the large-scale structure on a coarser grid
3) repeat steps 1-2 until reaching a minimum grid resolution
4) map to a finer grid (prolongation) and use as a guess for new iteration on the finer grid
5) repeat step 4) until reaching again the target grid resolution

Solving gravity: Multigrid Iterative Solvers

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Con ... :

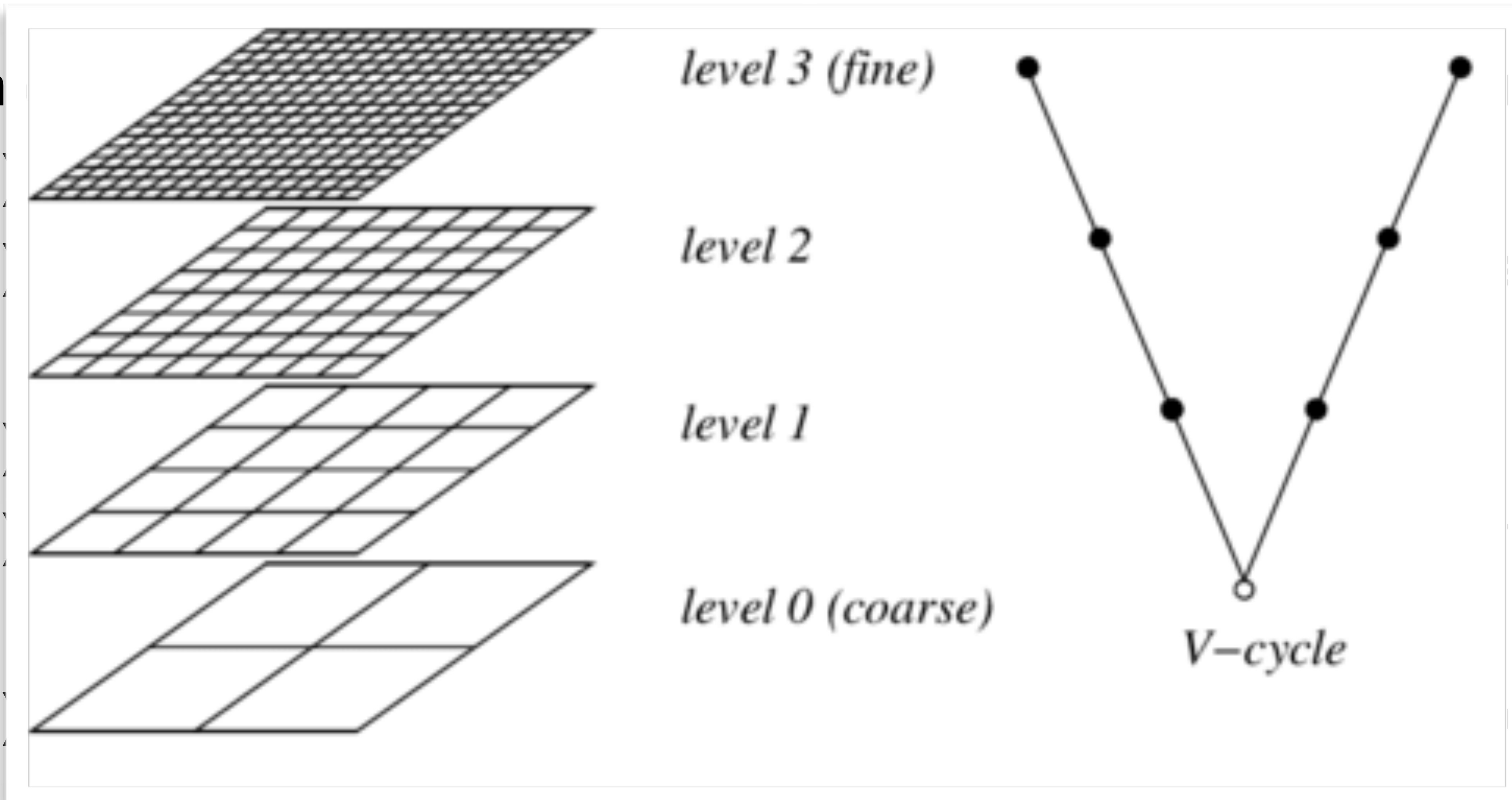
1)

2) ... rser

3)

4) ... ew

5) ... n



This is called V-cycle, and is shown to converge as $\mathcal{O}(N \log N)$

How?

Solving gravity: comparing solvers

Gravity solvers compared

Gravity solvers compared

Particle-Mesh

Solve

$$\nabla_x^2 \Phi = 4\pi G a^2 \rho(\mathbf{x})$$

by summing individual potential contributions over all volume elements

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

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which is a consequence of the linearity of Poisson Equation

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Does not rely on the superposition principle

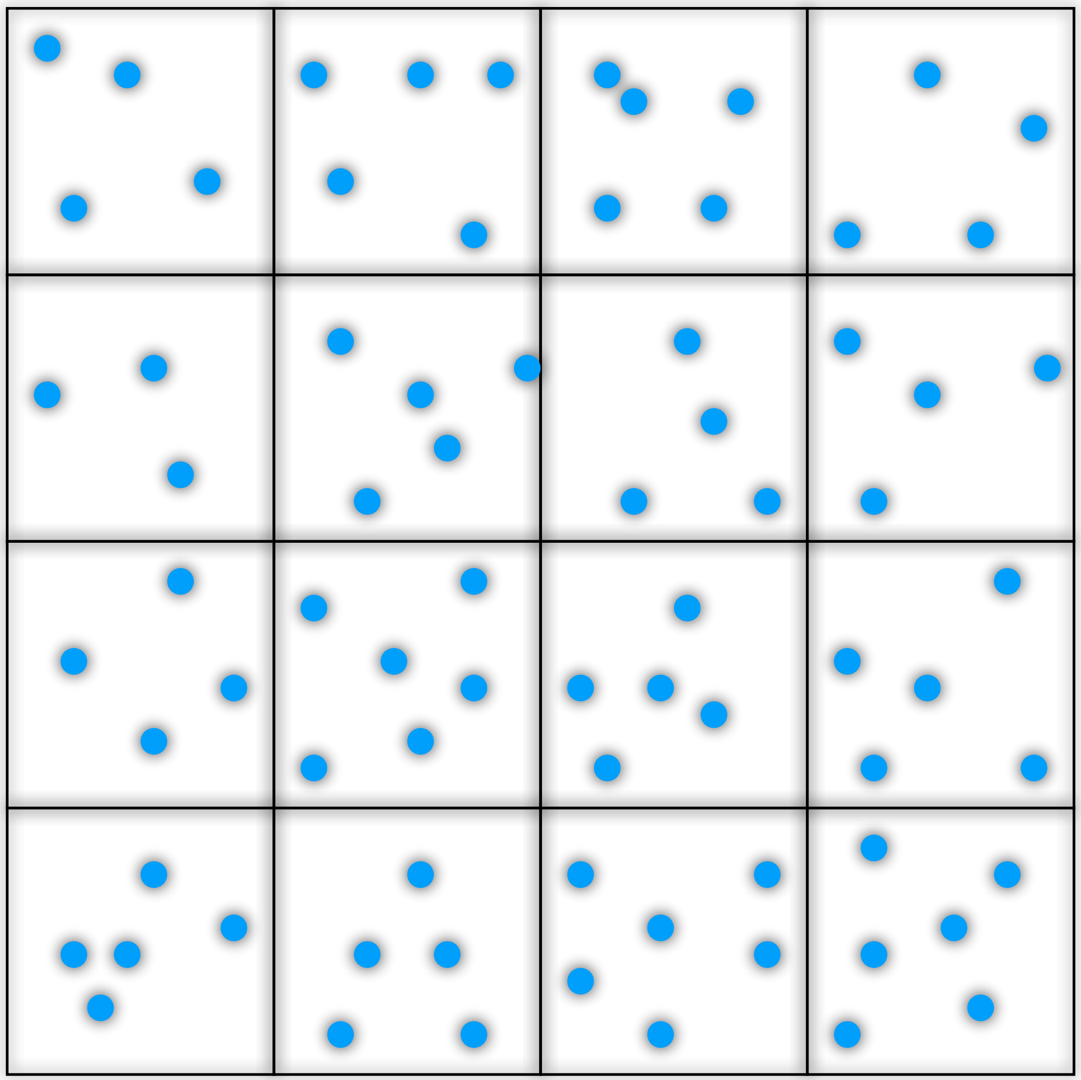
Works also for non-linear Poisson-like equations

How?

Time integration

Step 3: Time integration

Once the force (hence the acceleration) on each particle is known, the system has to be moved **forward in time (positions and velocities)**



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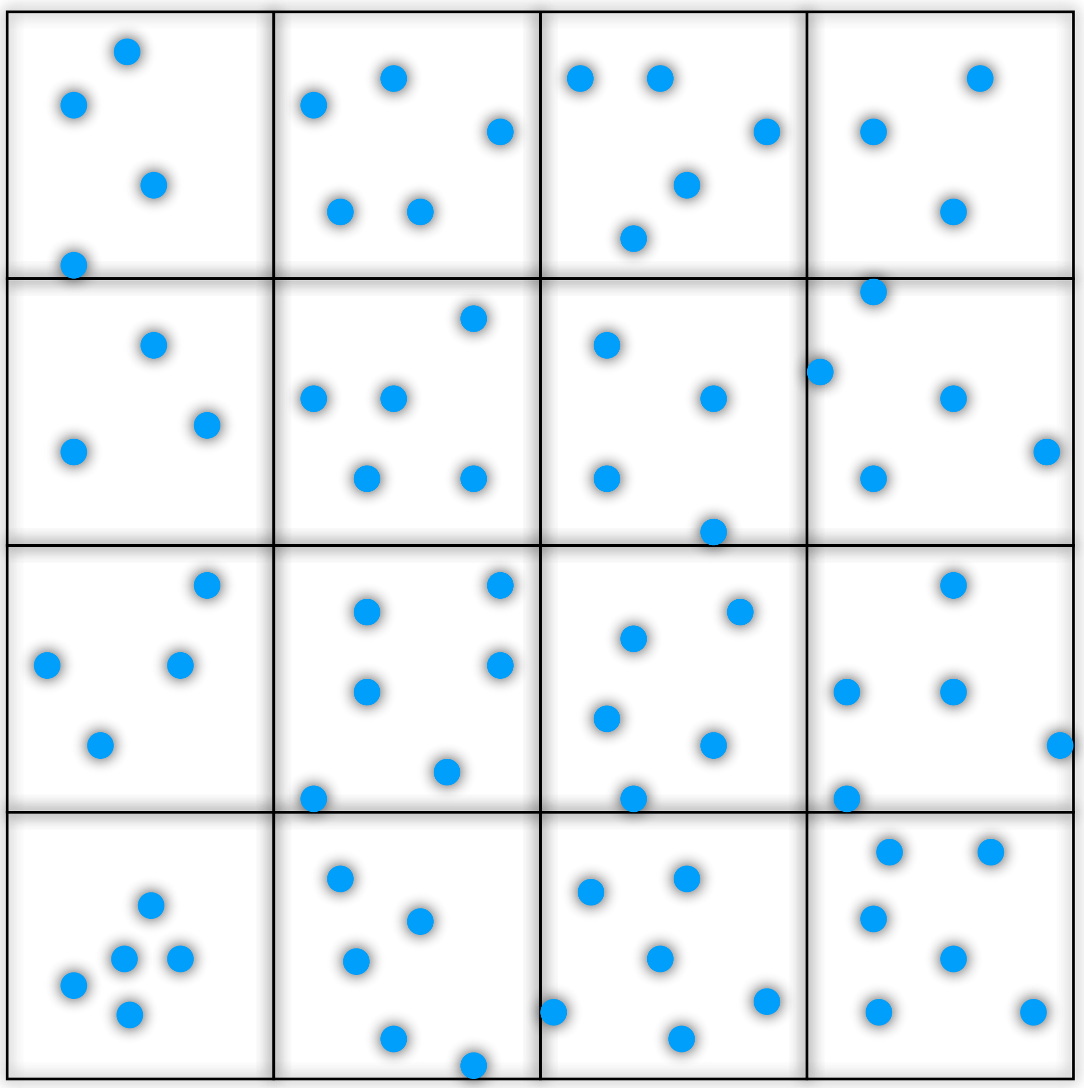

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Several possible time integration schemes (Euler, Runge-Kutta, mid-point), the most widely used is the **LEAPFROG:**

$$v_{n+\frac{1}{2}} = v_n + a_n \frac{\Delta t}{2} \qquad \text{Kick}$$

$$x_{n+1} = x_n + v_{n+\frac{1}{2}} \Delta t \qquad \text{Drift}$$

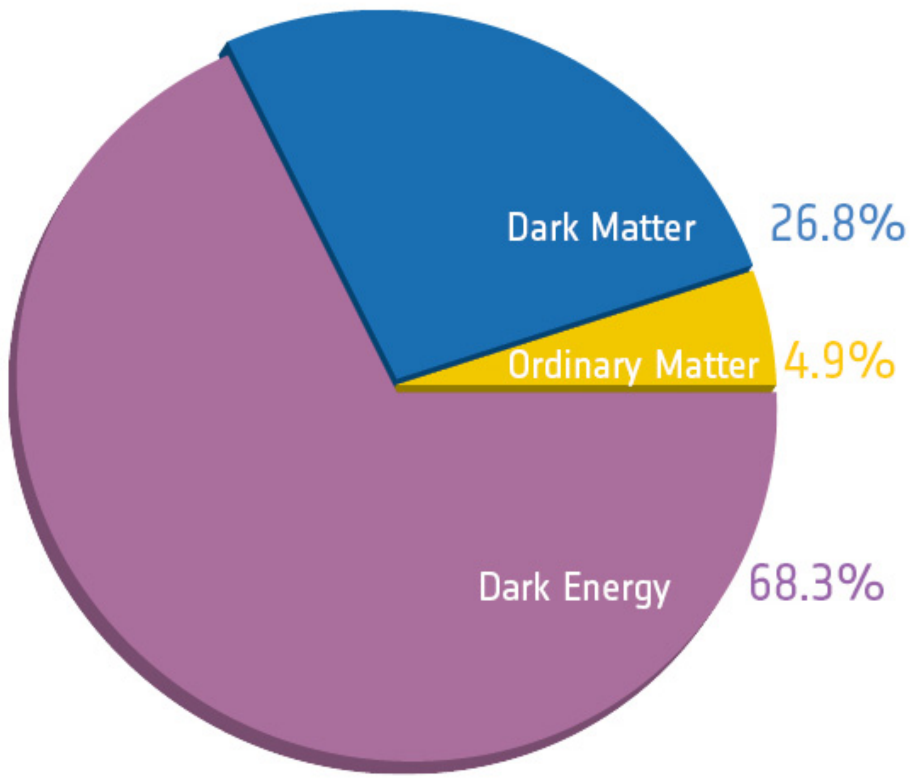
$$v_{n+1} = v_{n+\frac{1}{2}} + a_{n+1} \frac{\Delta t}{2} \qquad \text{Kick}$$

Time-reversible and explicitly energy conserving

What else?

Extending to non-standard models

Standard Cosmology after Planck



The Universe after Planck:
6 parameters to fit all data

$\Omega_{\rm b} = 0.049$ $\quad \Omega_{\rm CDM} = 0.265$

$\Omega_\Lambda = 0.6844$ $\quad \sigma_8 = 0.831$

$n_s = 0.9645$ $\quad \tau = 0.079$

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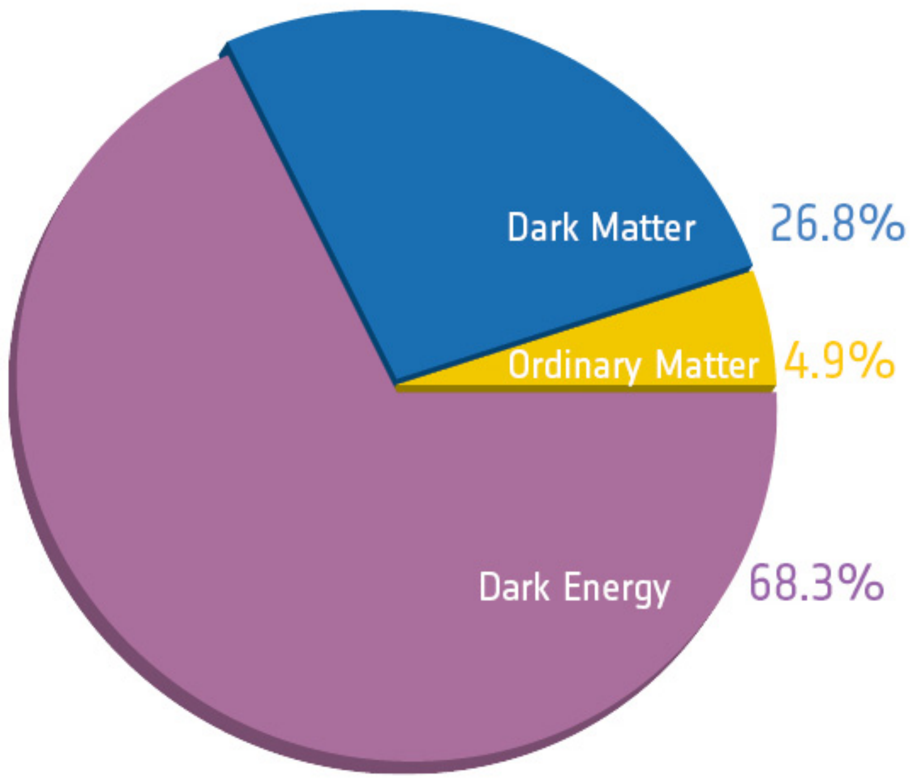

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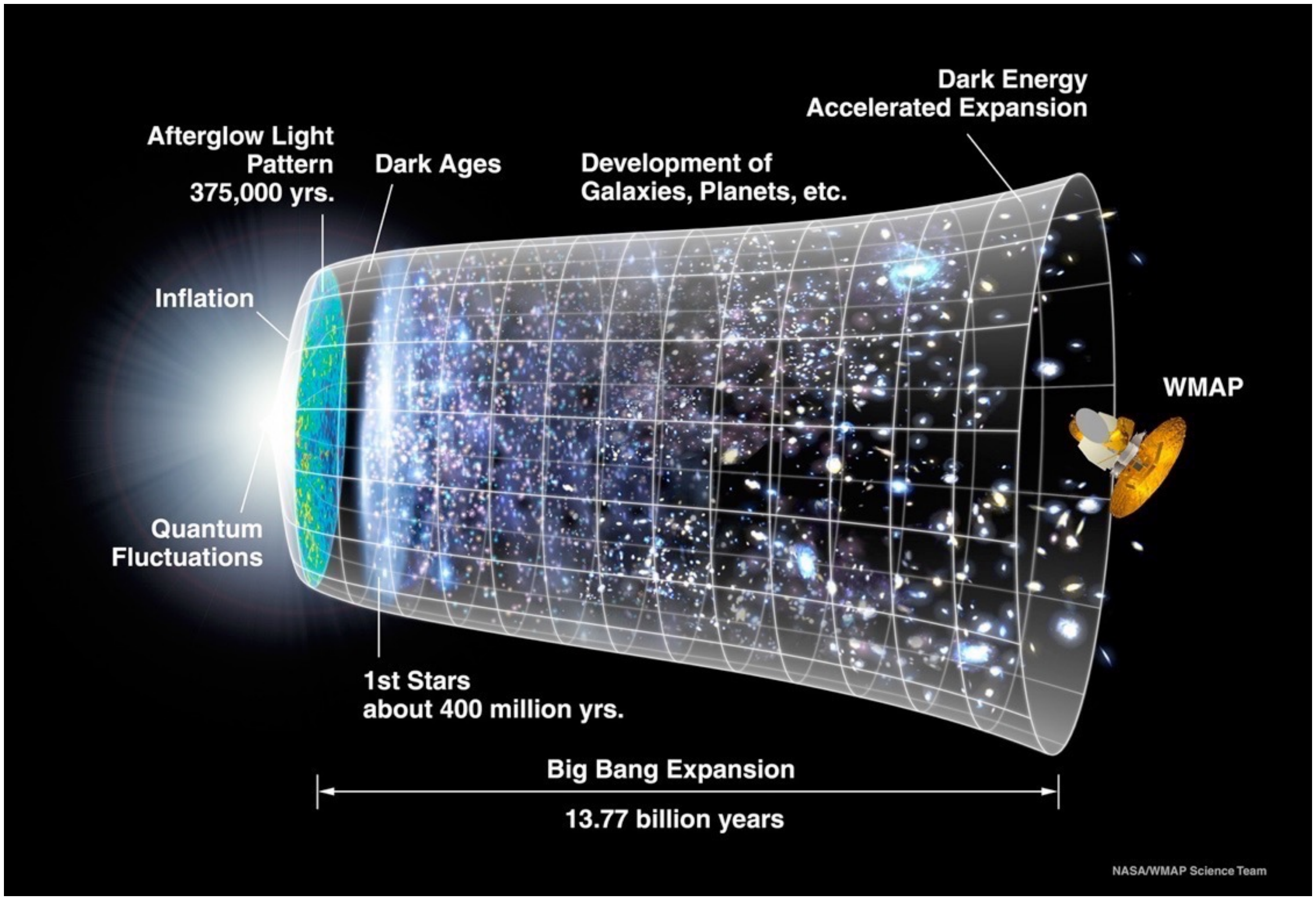
Standard ΛCDM cosmology is based on a series of assumptions:

- Cosmological Principle (homogeneity & isotropy);
- Gaussian and Adiabatic initial conditions;
- Dark Matter is Cold and Collisionless;
- Neutrinos are massless;
- Dark Energy is a Cosmological Constant;
- GR is the complete theory of gravity;

The Dark Energy problem

Starting point:

observational evidence of an accelerated expansion



Can we explain this somehow?

The Dark Energy problem

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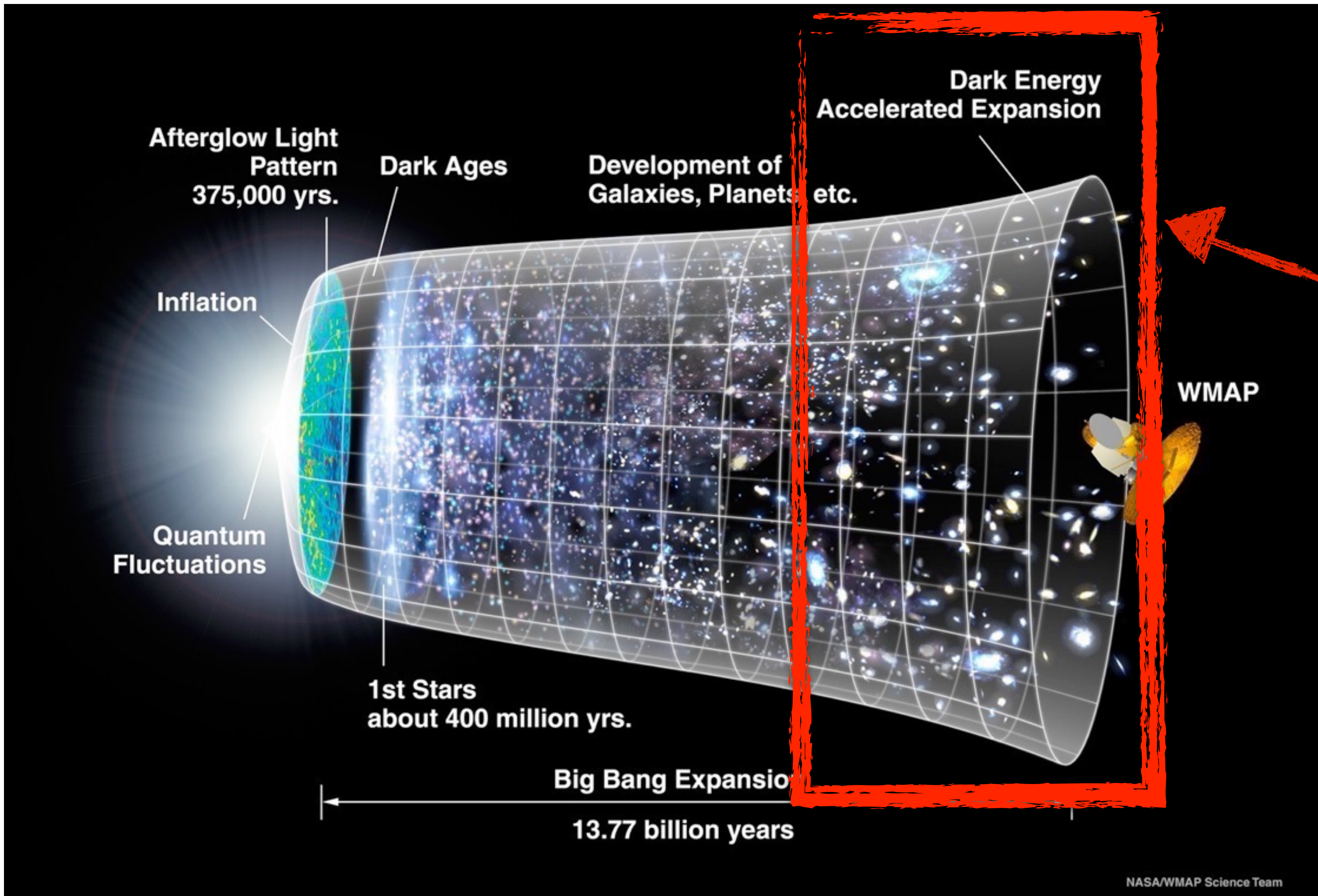

"Until it is solved, the problem of the dark energy will be a roadblock on our path to a comprehensive physical theory"

S. Weinberg

Can we explain this somehow?

Motivations for beyond-Lambda models

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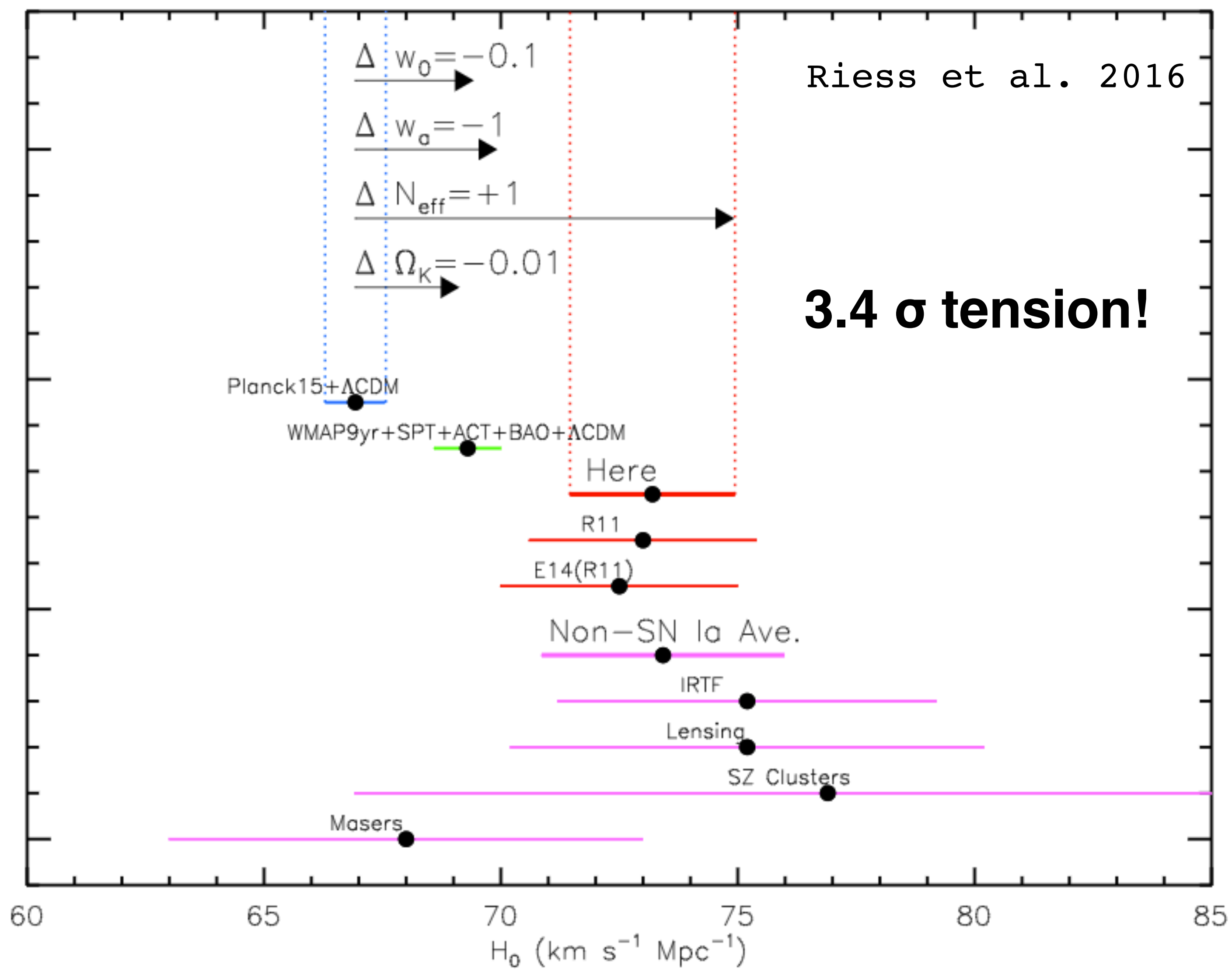

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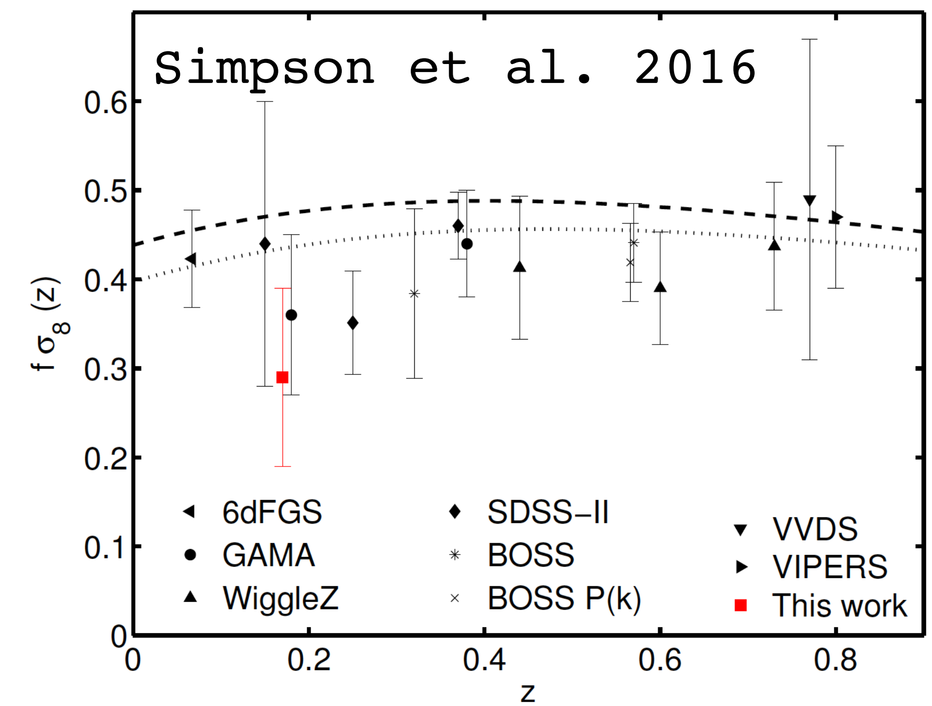
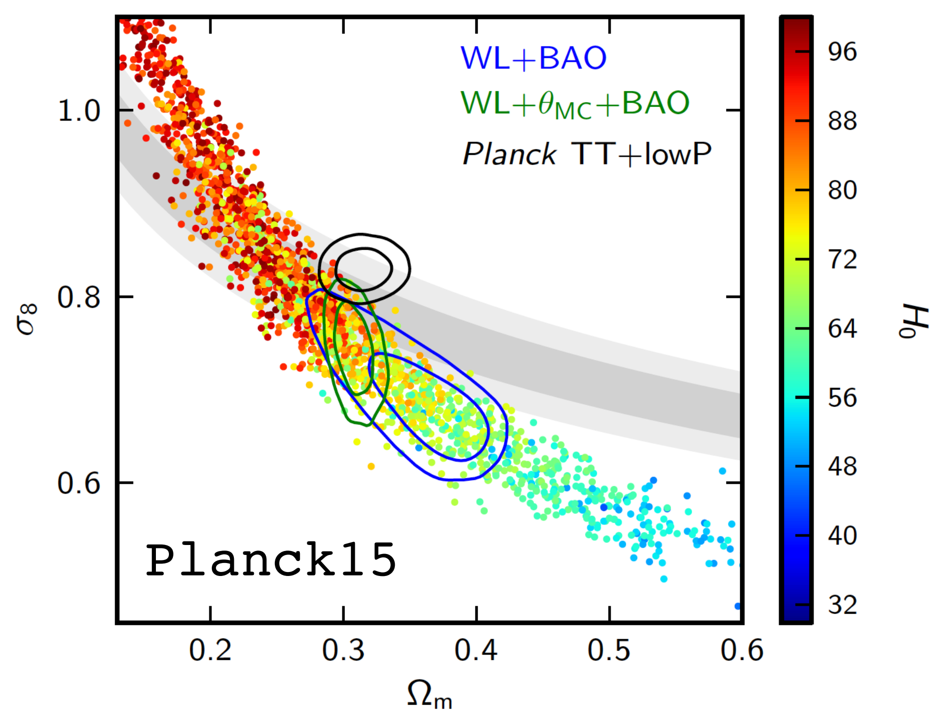
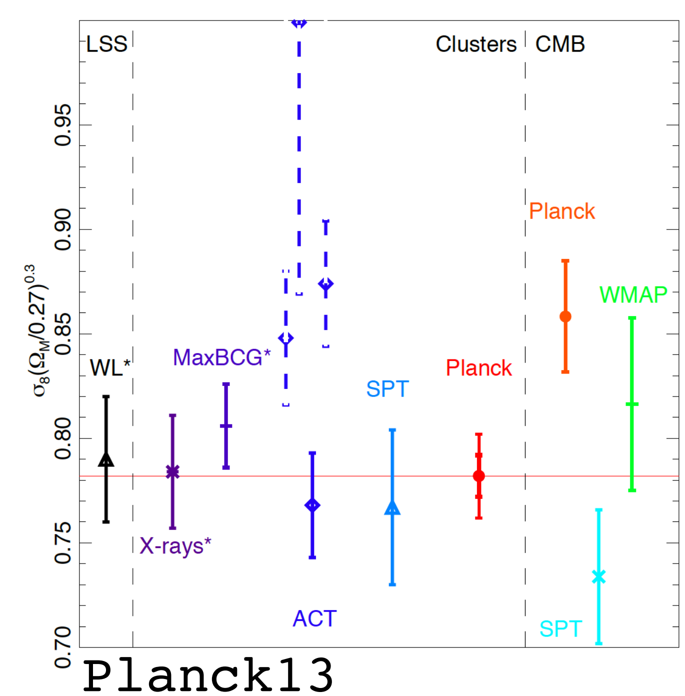
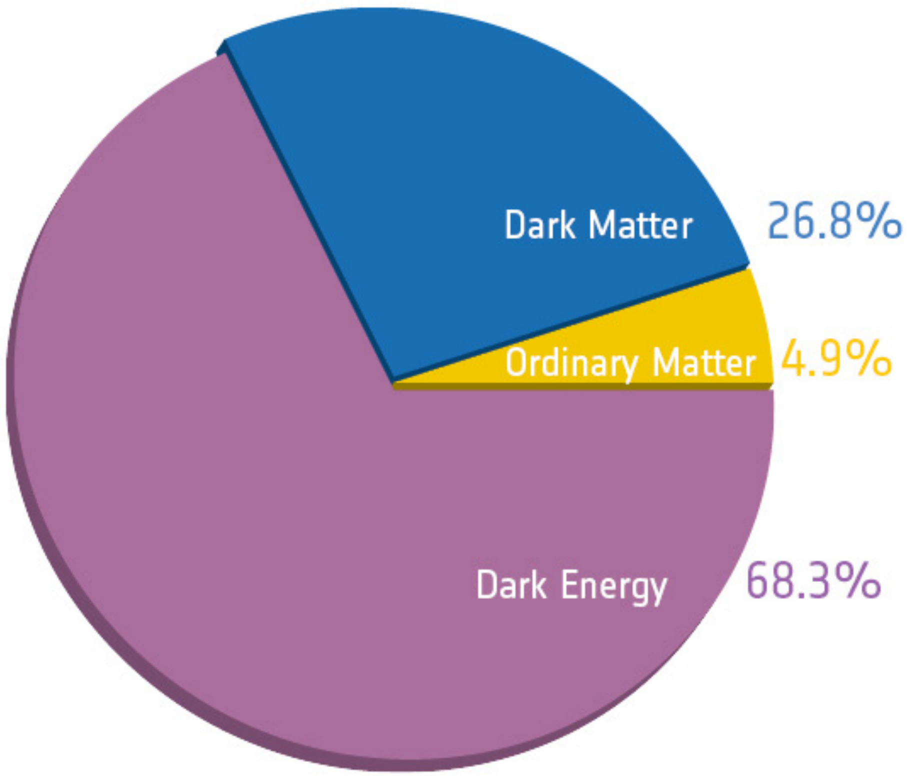






Testing the assumptions of the Standard Model



The Universe after Planck: 6 parameters to fit all data

$$\Omega_b = 0.049 \qquad \Omega_{CDM} = 0.265$$

$$\Omega_\Lambda = 0.6844 \qquad \sigma_8 = 0.831$$

$$n_s = 0.9645 \qquad \tau = 0.079$$

Standard ΛCDM cosmology is based on a series of assumptions:

- Cosmological Principle;
- Gaussian and Adiabatic initial conditions;
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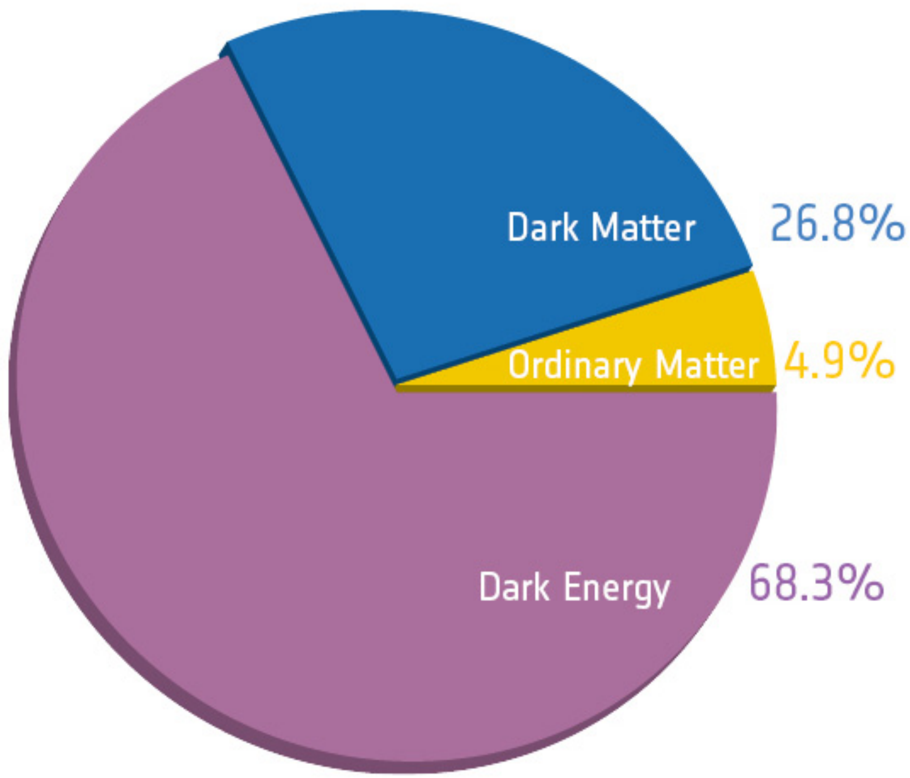

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DARK ENERGY MODELS BEYOND THE COSMOLOGICAL CONSTANT

Classification of Dark Energy models

This will be just a purely phenomenological classification, based on the expected effects on structure formation and how these will affect N-body simulation methods.

For a more formal classification, see e.g.
Pourtsidou et al. (2013)
Skordis et al. (2015)

Classification of Dark Energy models

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Interacting DE (Coupled and Extended Quintessence, Modified Gravity)	✔ a dynamical (scalar) degree of freedom	✔ fluctuations sourced by the interaction	✔ non-minimally coupled to matter

Interacting Dark Energy and structure formation

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An interaction between a matter field and a scalar field can be described by a source term in the respective continuity equations:

$$\nabla_\mu T^{\mu(\phi)}_\nu = -QT^{(\mathrm{DM})}\nabla_\nu\phi \qquad \nabla_\mu T^{\mu(\mathrm{DM})}_\nu = +QT^{(\mathrm{DM})}\nabla_\nu\phi$$

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By perturbing these equations at linear order one obtains a dynamical equation for the field perturbations:

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that in the so-called quasi-static limit ($\partial^2\delta\phi/\partial t^2 \ll \nabla^2\delta\phi$) gives:

$$\nabla^2\delta\phi = -\frac{dV}{d\phi}(\delta\phi) - Q\delta_{\rm DM}$$

and assuming a flat potential ($dV/d\phi \ll \delta_{\rm DM}$):

$$\nabla^2\delta\phi \approx -Q\delta_{\rm DM} \Rightarrow \delta\phi \approx -Q\Phi$$

Interacting Dark Energy and structure formation

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By combining this relation with the linear perturbations equations of the matter field (Amendola 2004), one obtains a modified version of the gravitational instability equation:

$$Q = 0 \quad \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega\delta = 0$$

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Scalar fifth-forces are tightly constrained from solar system tests of gravity, need to allow for a non-universal coupling so that CDM is coupled and baryons are uncoupled ($Q_c \neq 0,\ Q_b = 0$):

$$\ddot{\delta}_c + \left(2H - 2Q\dot{\phi}\right)\dot{\delta}_c - \frac{3}{2}H^2\left[\left(1 + 2Q^2\right)\Omega_c\delta_c + \Omega_b\delta_b\right] = 0$$

$$\ddot{\delta}_b + 2H\dot{\delta}_b - \frac{3}{2}H^2\left[\Omega_c\delta_c + \Omega_b\delta_b\right] = 0$$

Interacting Dark Energy and structure formation

This implies an effective violation of the Weak Equivalence Principle:

$$\vec{a}_{\rm CDM} = -\vec{\nabla}\Phi(1+2Q^2)+2Q\dot{\phi}\vec{v}_{\rm CDM} \qquad \vec{a}_{\rm b} = -\vec{\nabla}\Phi$$

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The term $2Q\dot{\phi}\vec{v}_{\rm CDM}$ is called "friction term" and arises from momentum conservation:

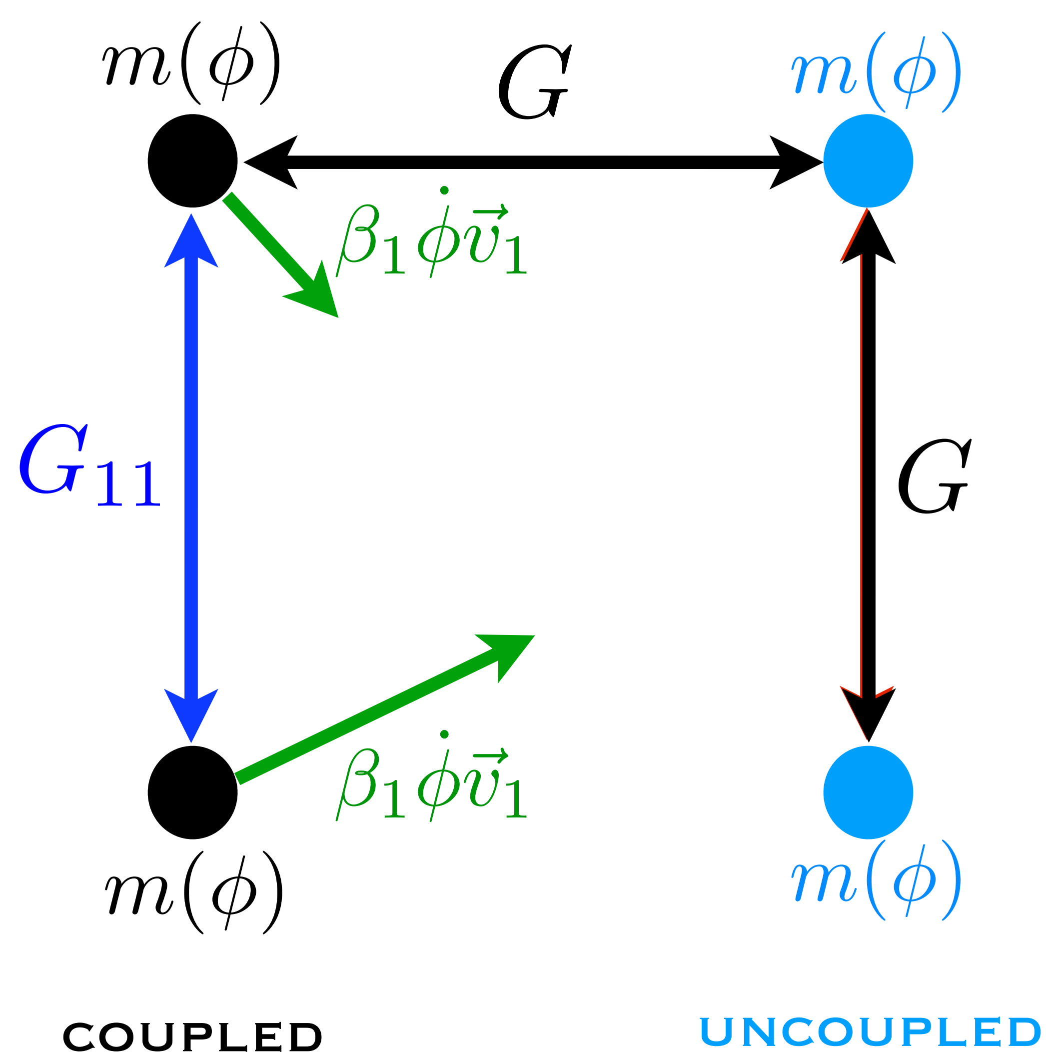
$$\frac{d\vec{p}}{dt} = \frac{d(m(\phi)\vec{v})}{dt} = m(\phi)\vec{a}+\frac{dm}{d\phi}\dot{\phi}\vec{v}$$

Interacting Dark Energy and structure formation

Standard Coupled Quintessence: interacting DM and non-interacting baryons:

$$\vec{a}_{\rm DM} = -\vec{\nabla}\Phi_{\rm N} - 2Q^2\vec{\nabla}\Phi_{\rm DM} + Q\dot{\phi}\vec{v}_{\rm DM}$$

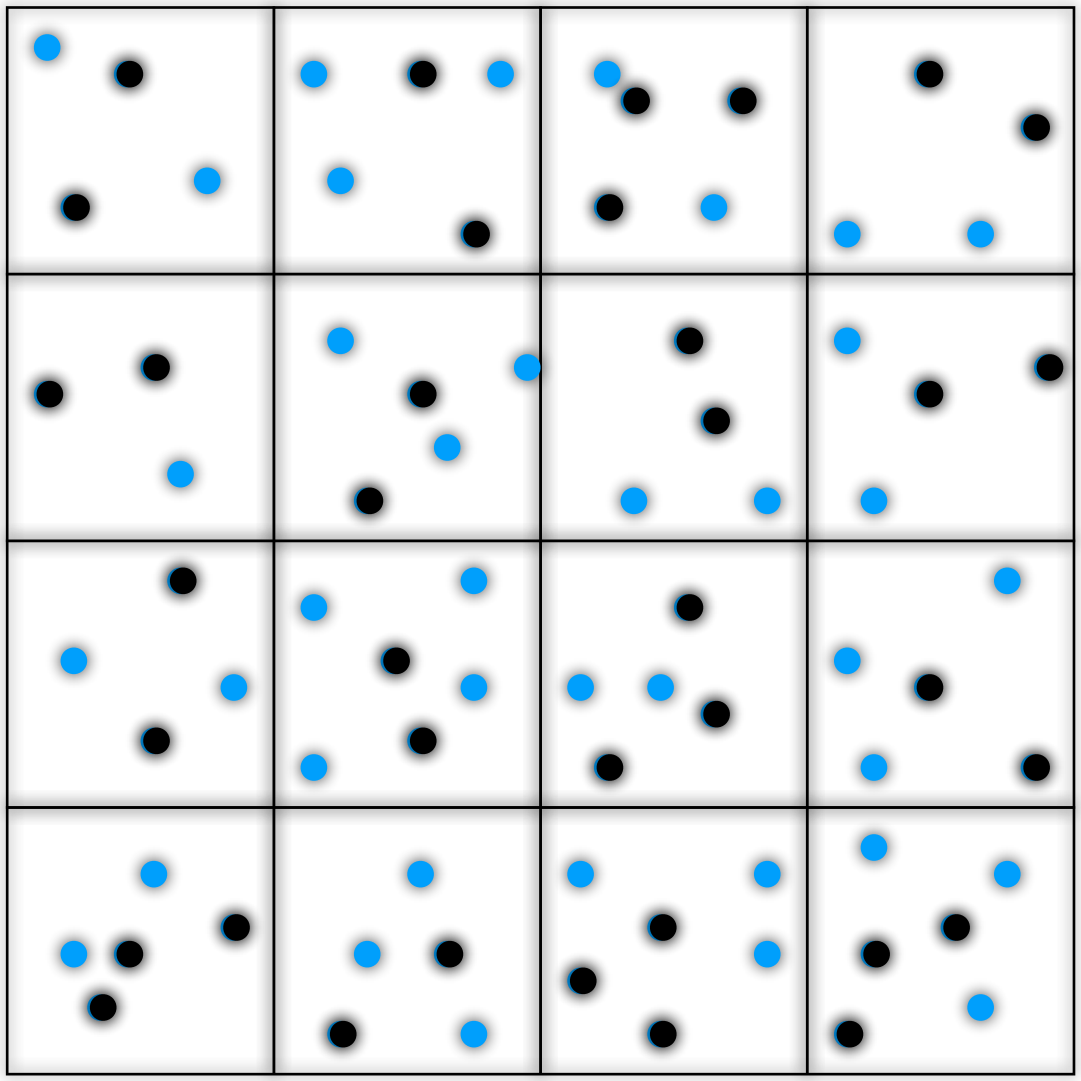
$$\vec{a}_{\rm b} = -\vec{\nabla}\Phi_{\rm N}$$



N-body algorithms for interacting Dark Energy

In interacting DE the coupling determines two different gravitational forces for dark matter and baryons:

Particle-Mesh (for interacting DE)



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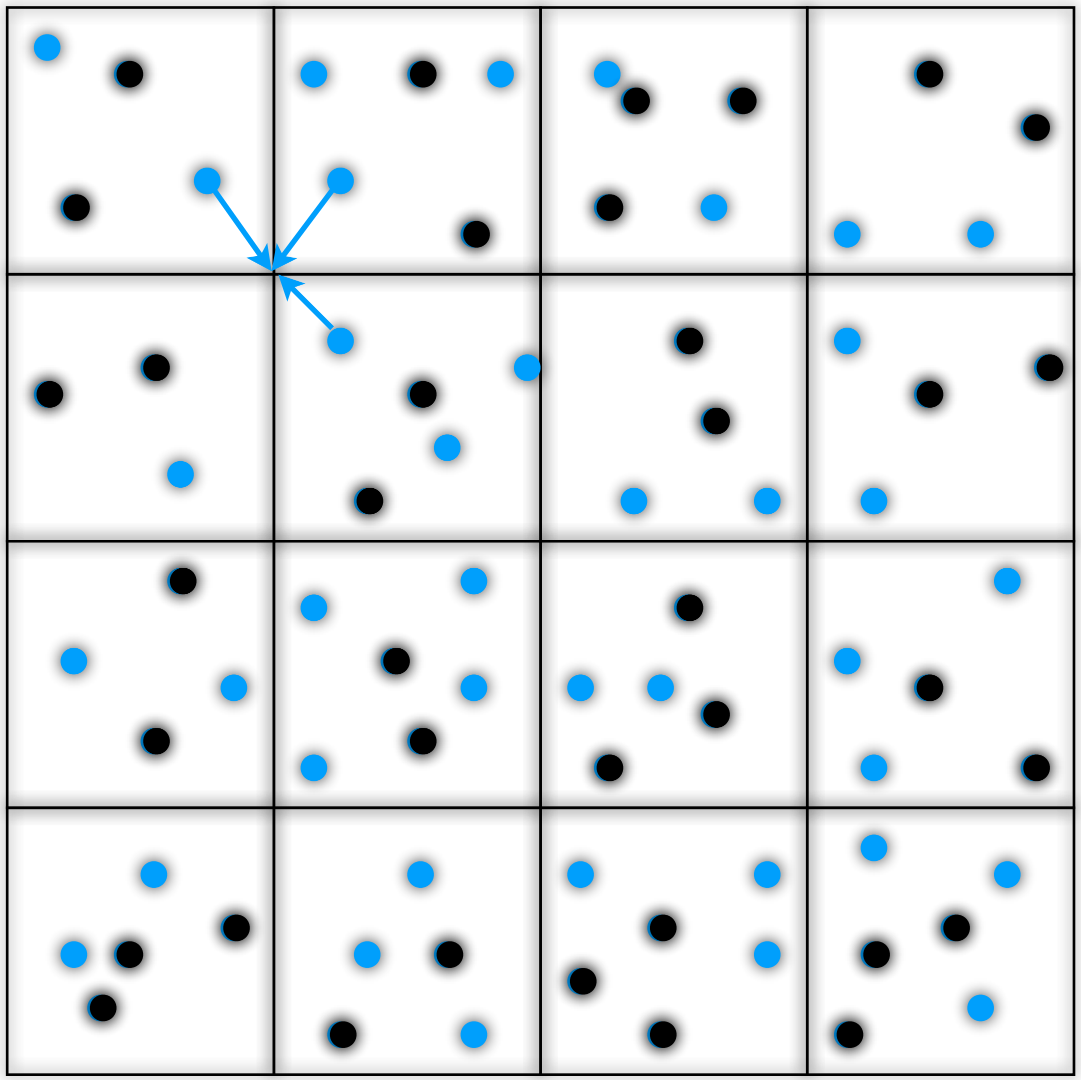

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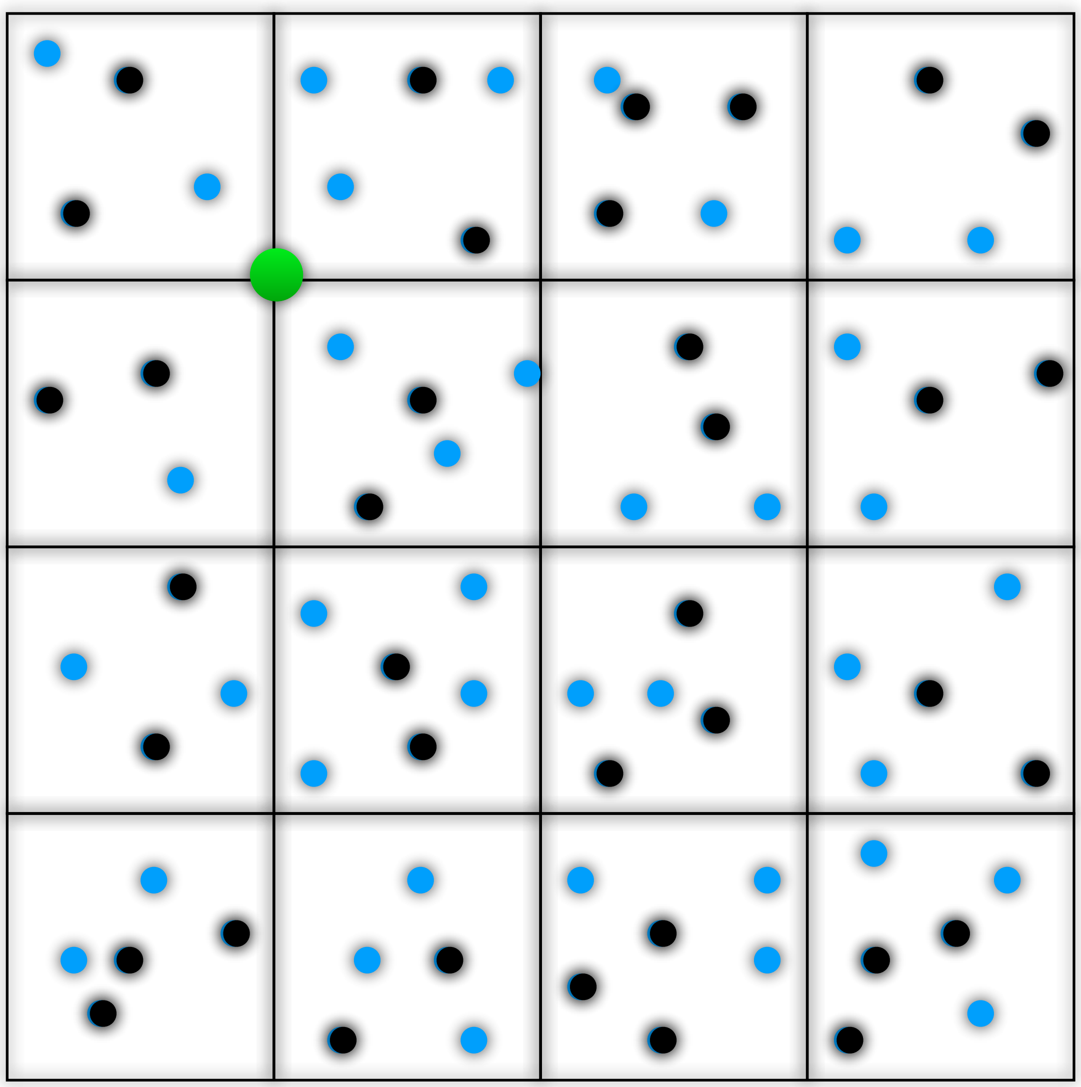
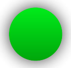


1) Assign mass to grid nodes for one species only, obtain density on the grid 

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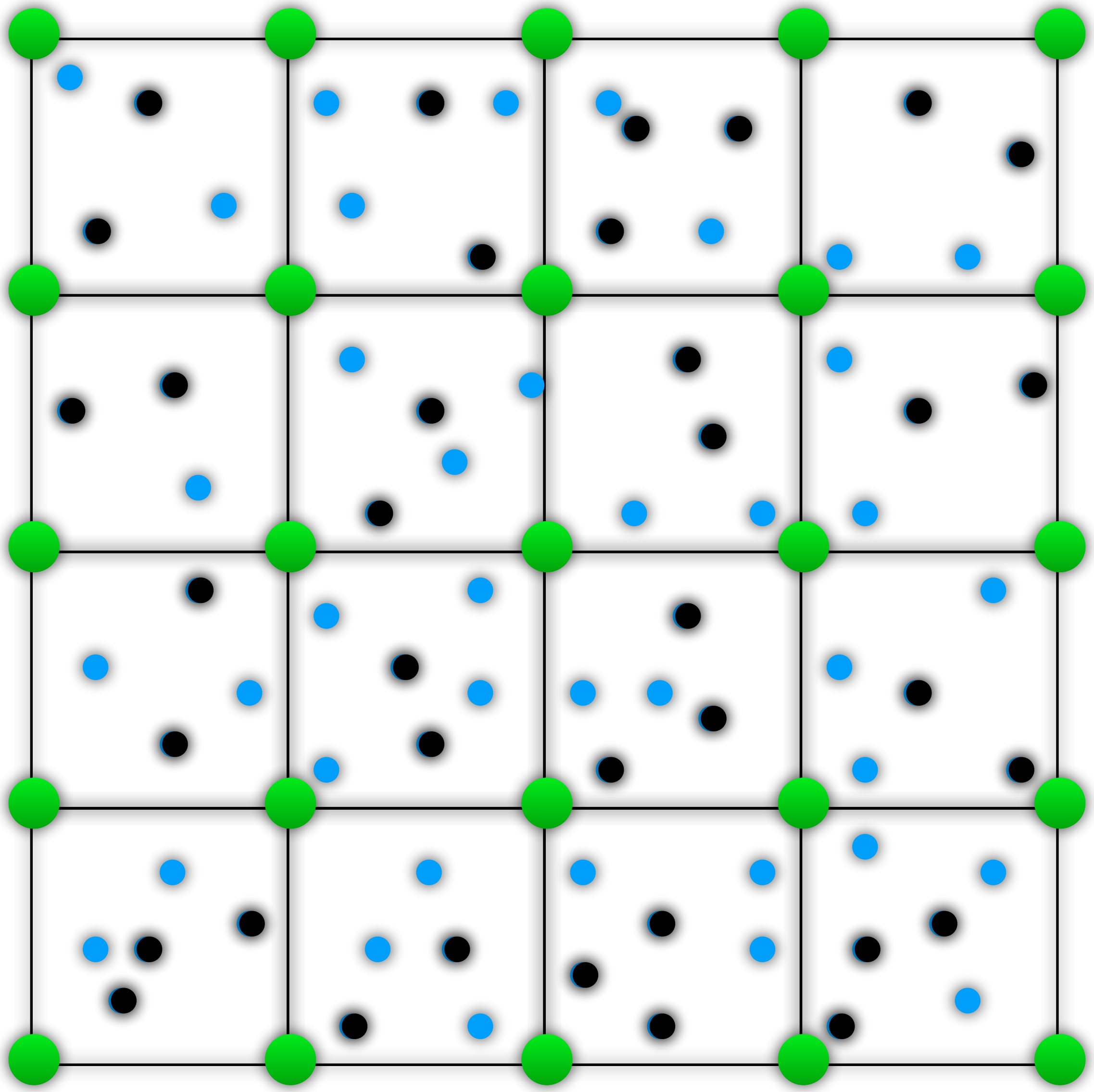

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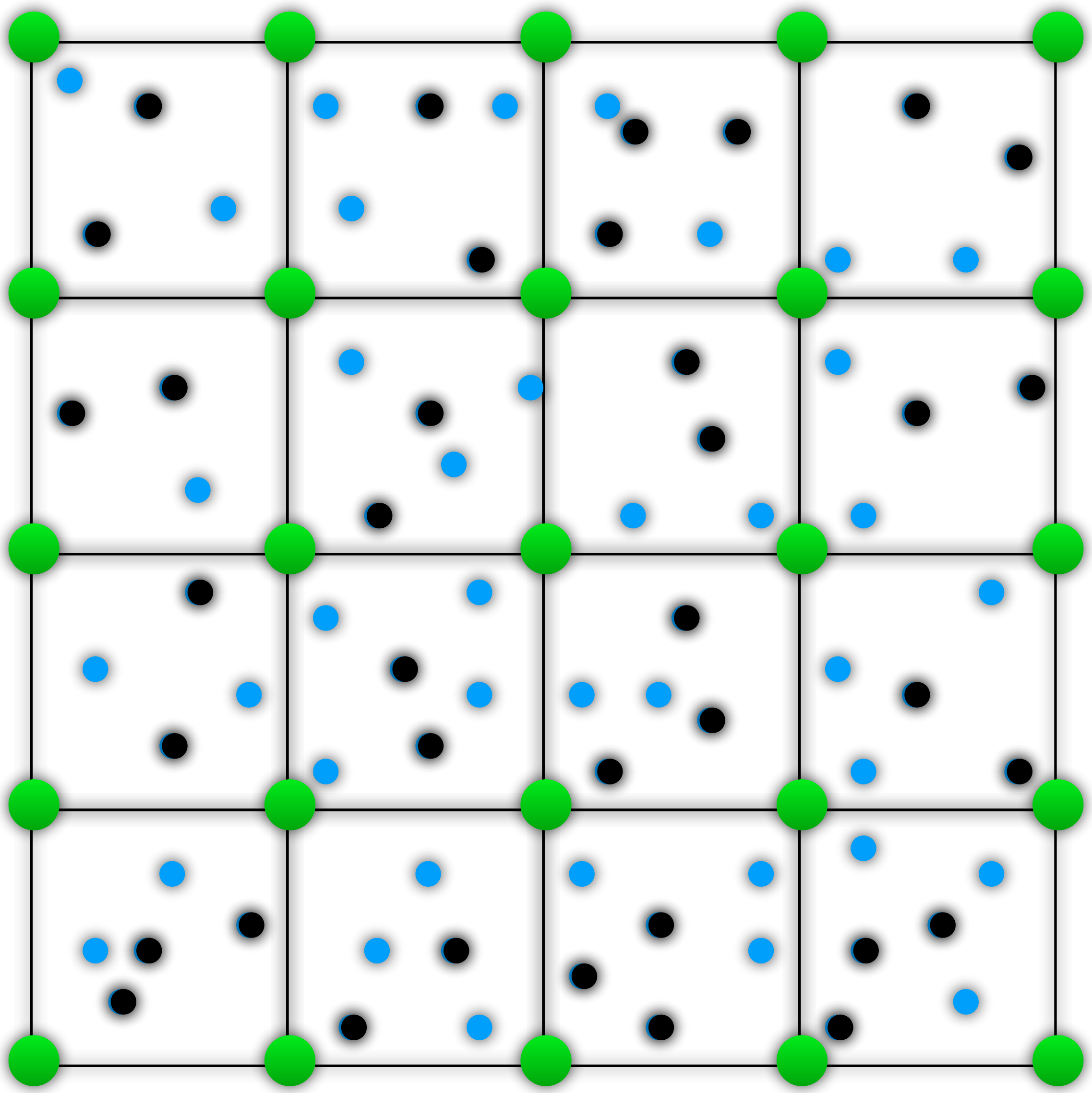

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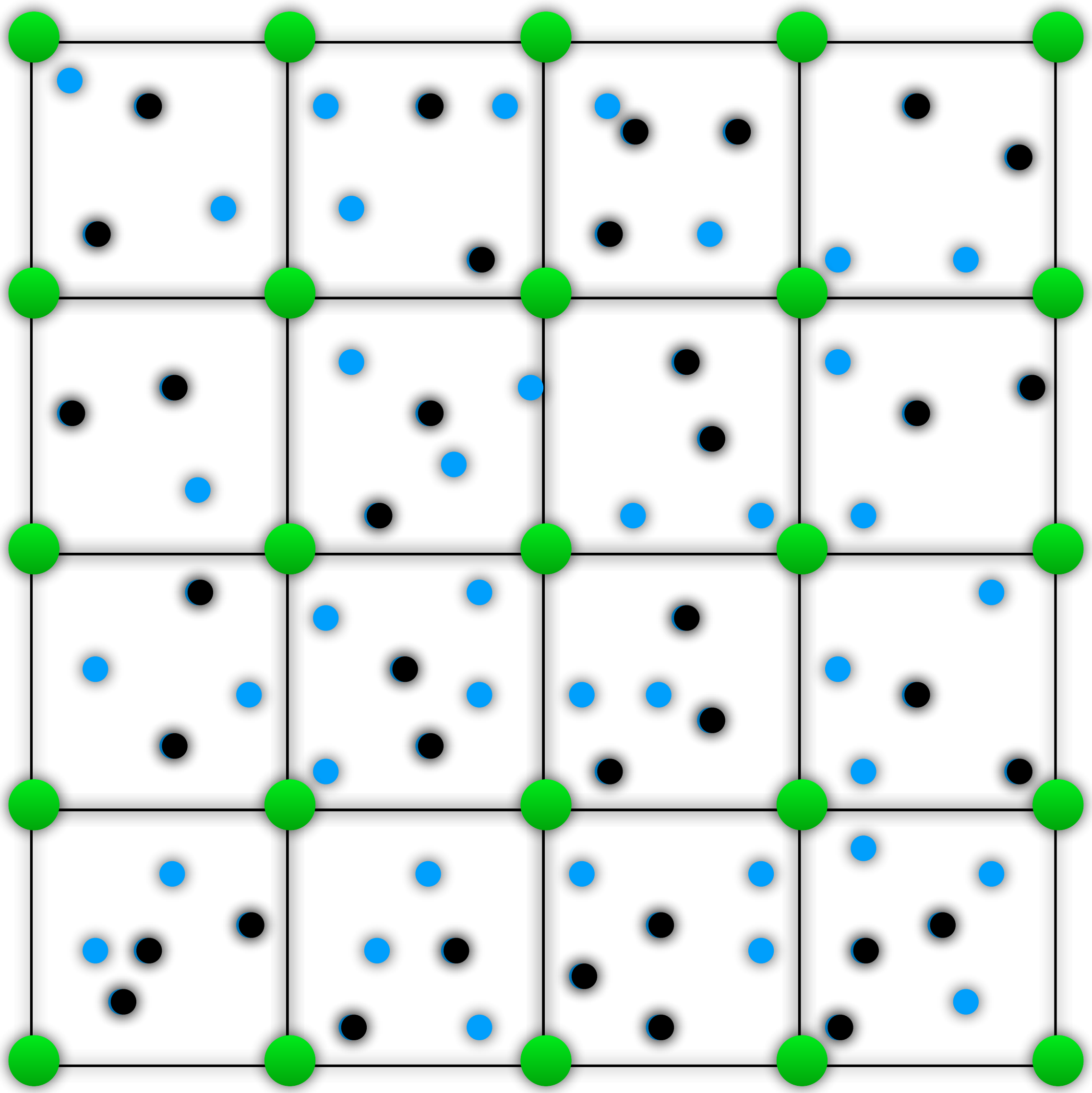

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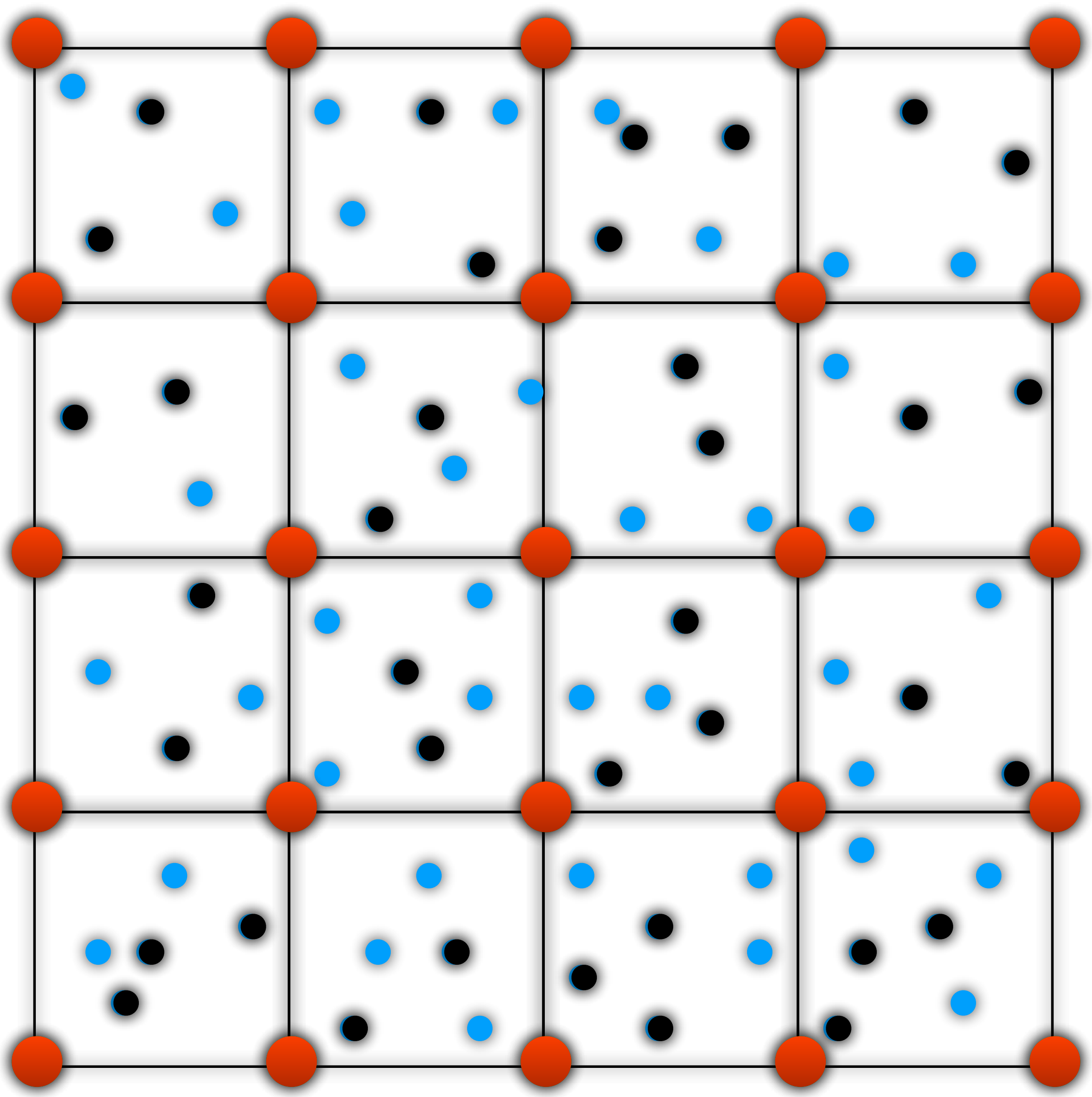

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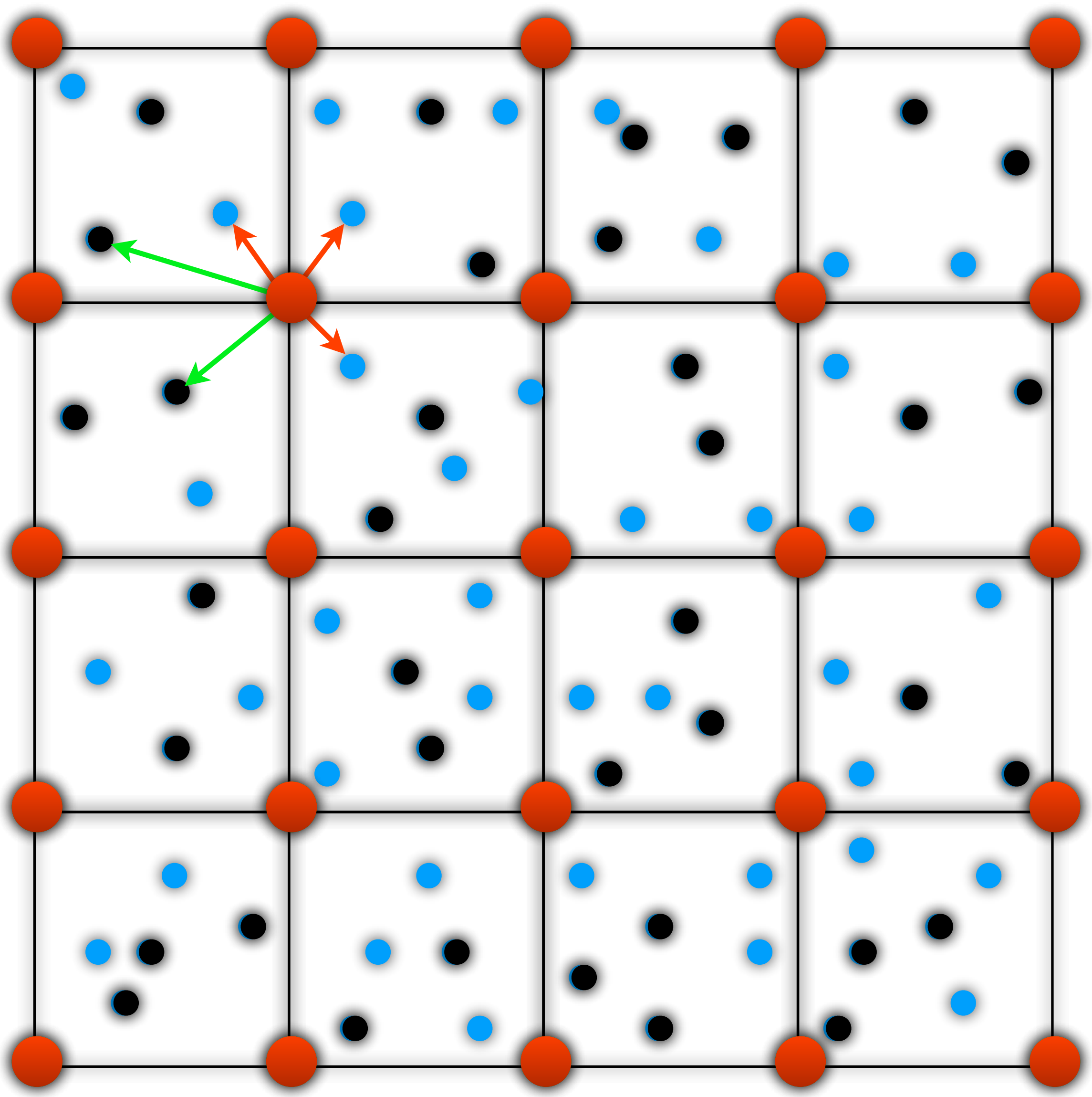

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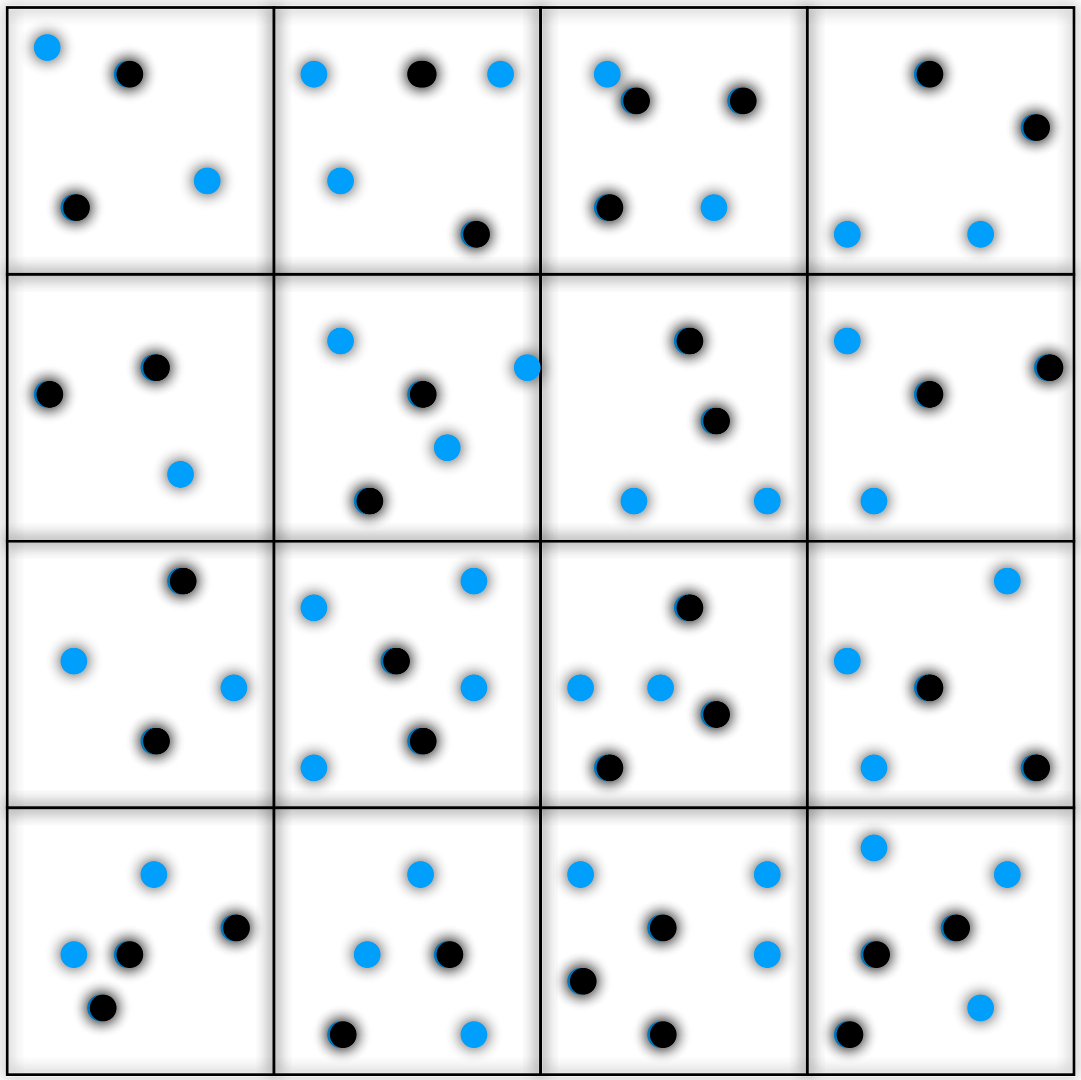

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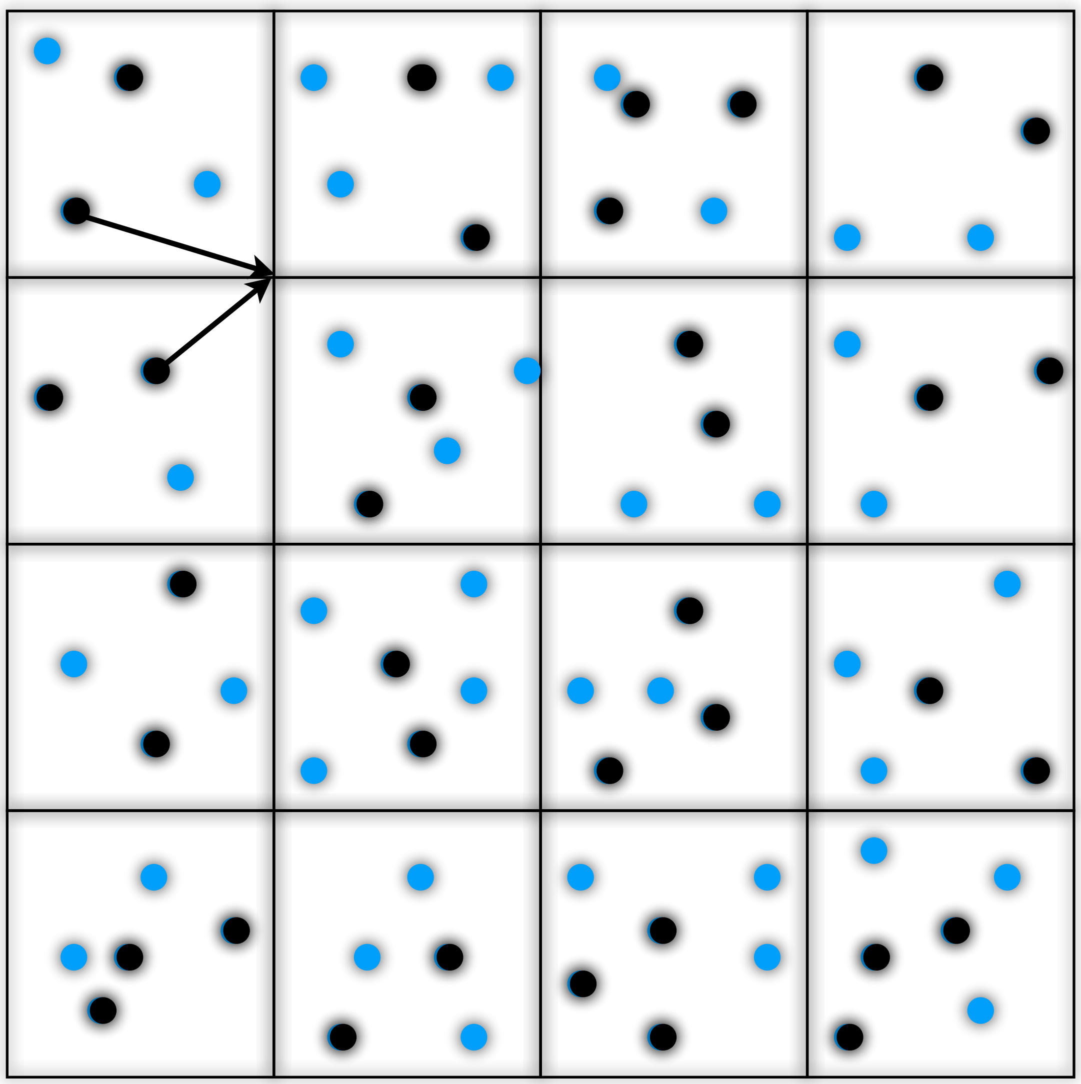

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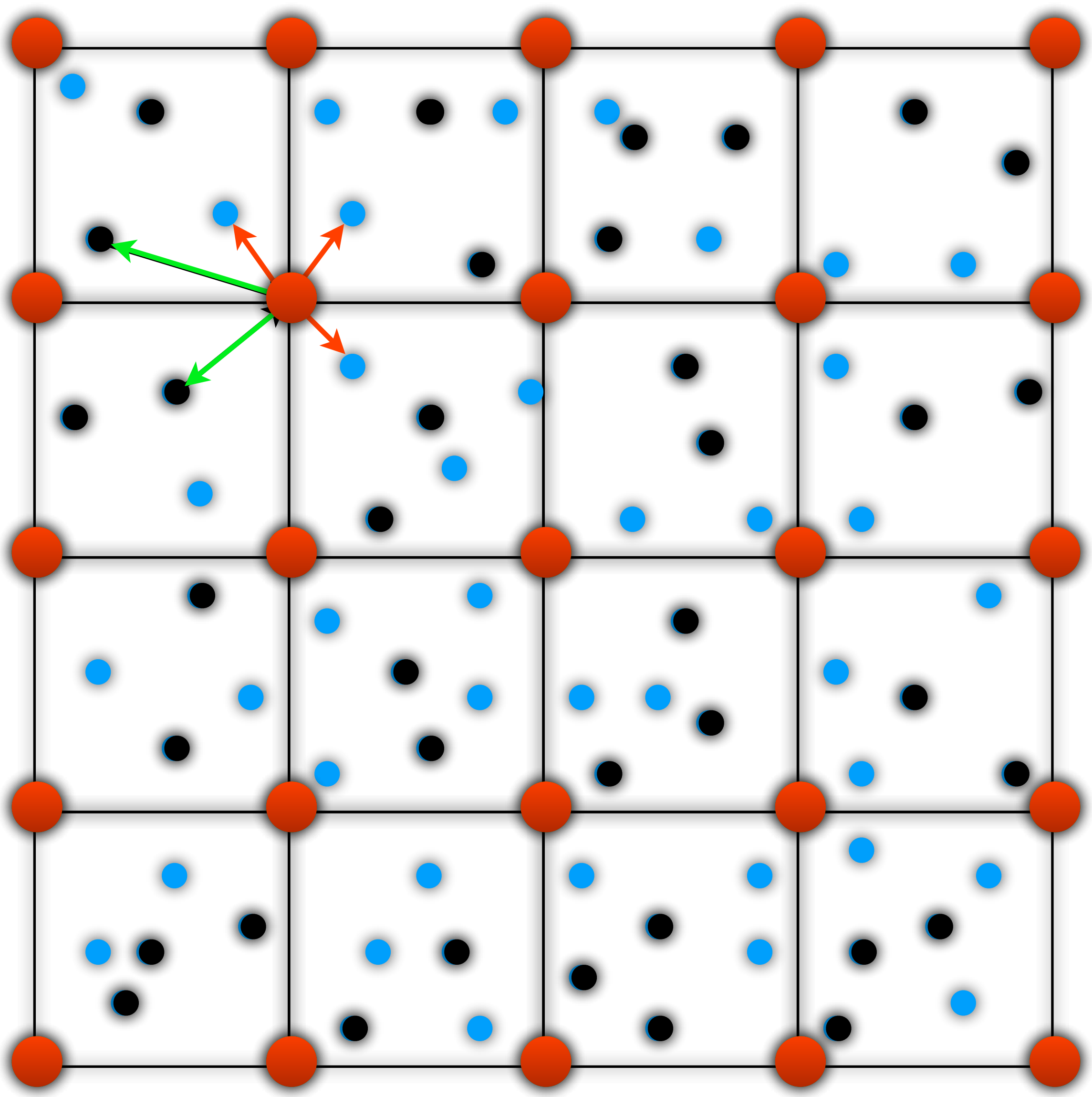

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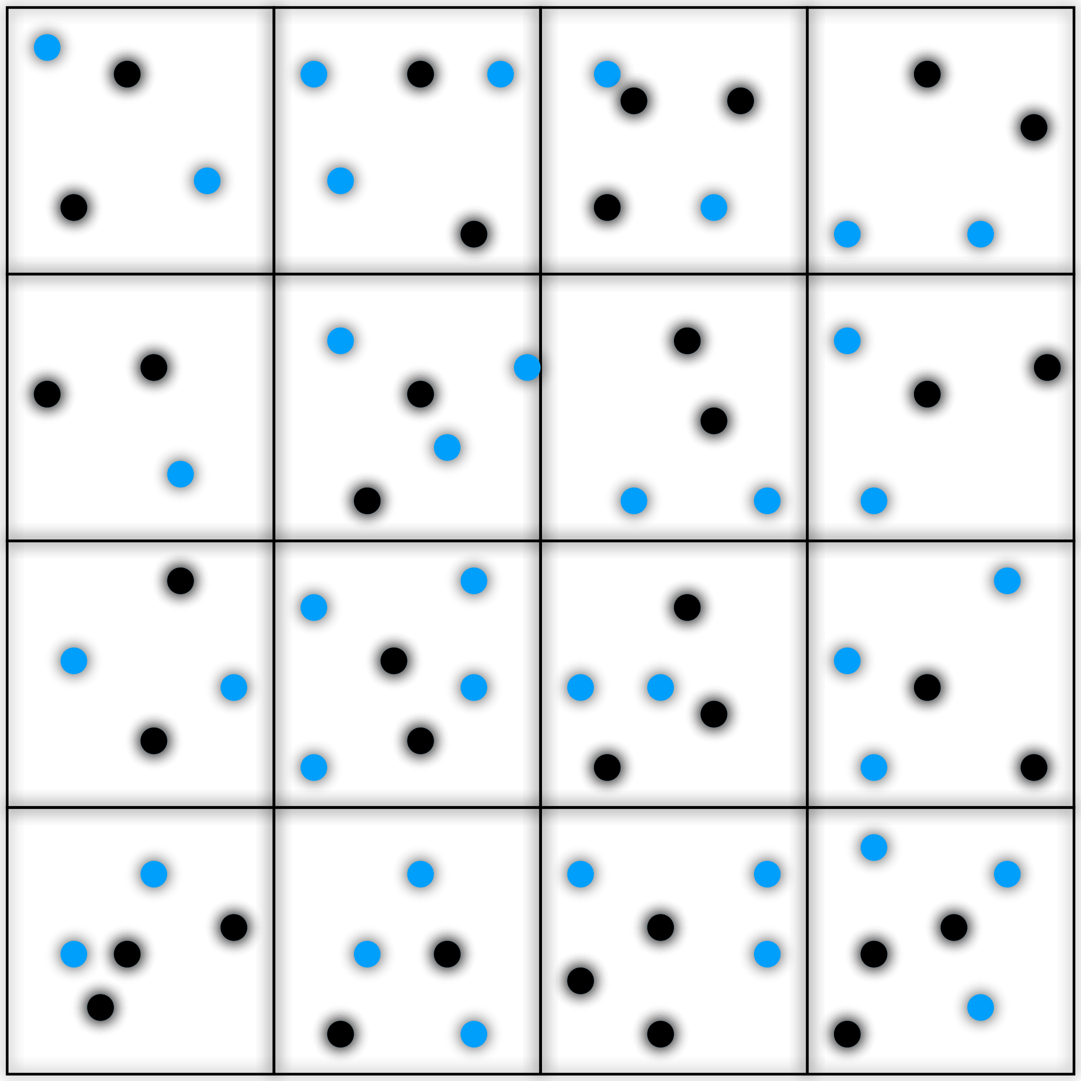

Compute the direct gravitational interaction of a TARGET particle with another particle or a NODE (group of particles)

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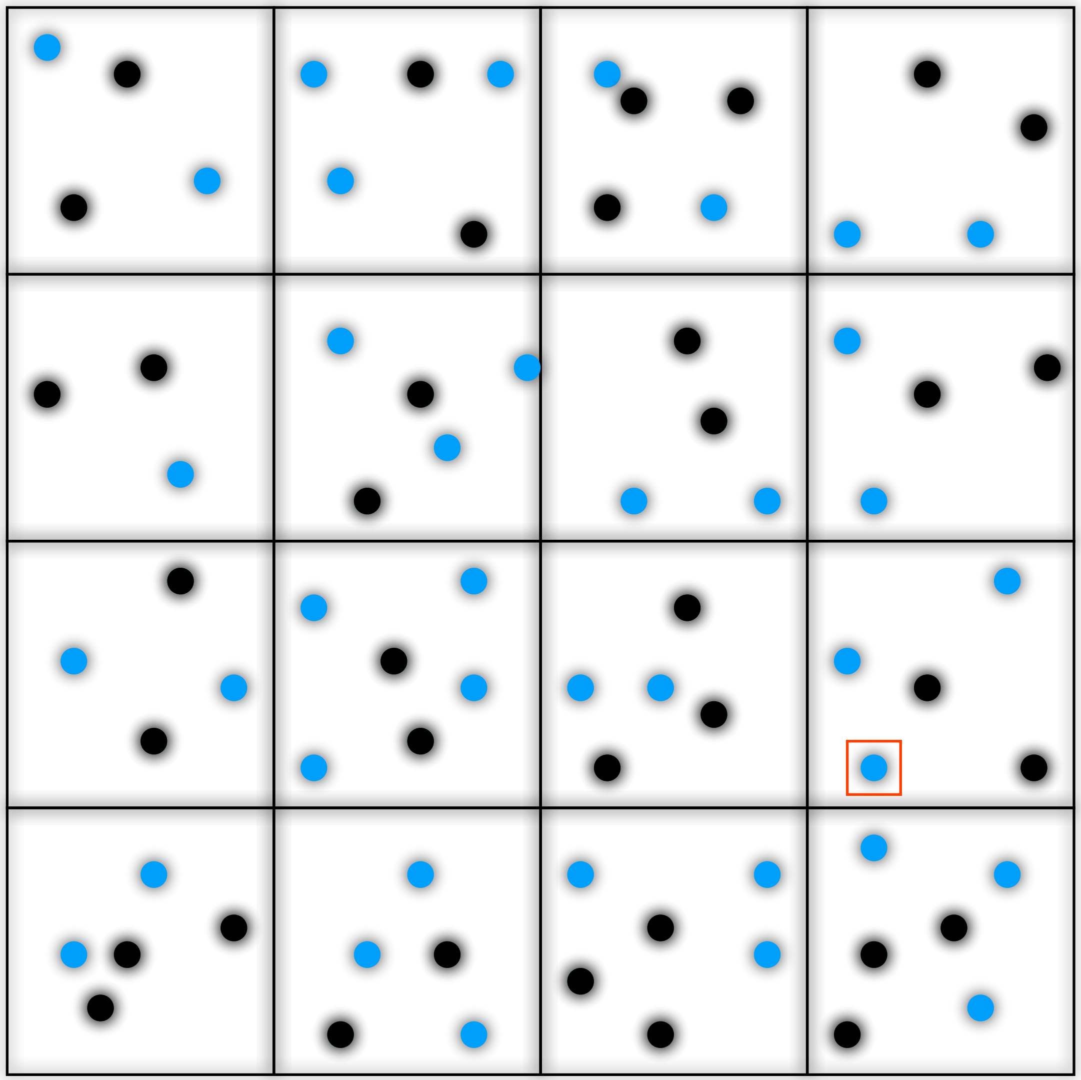

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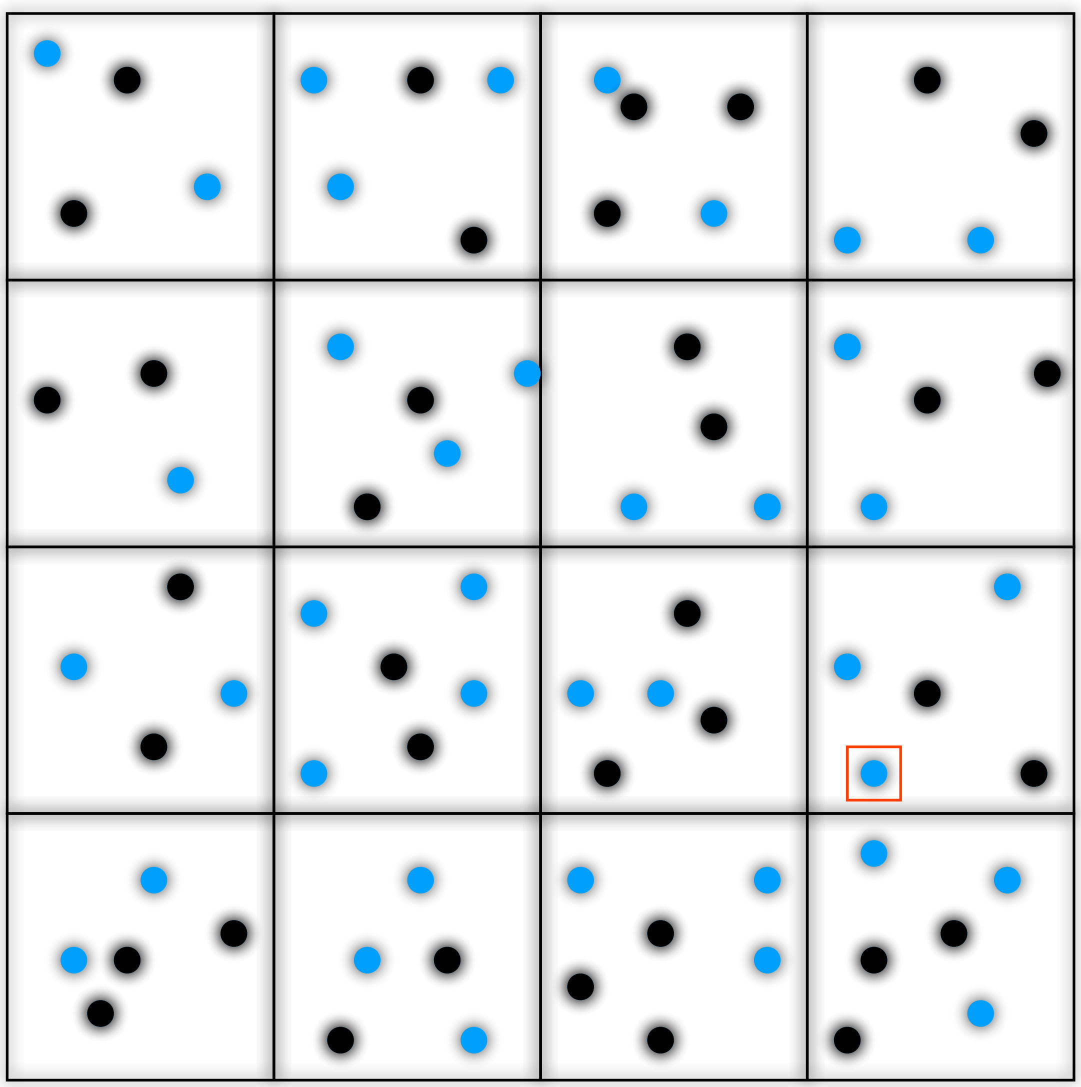

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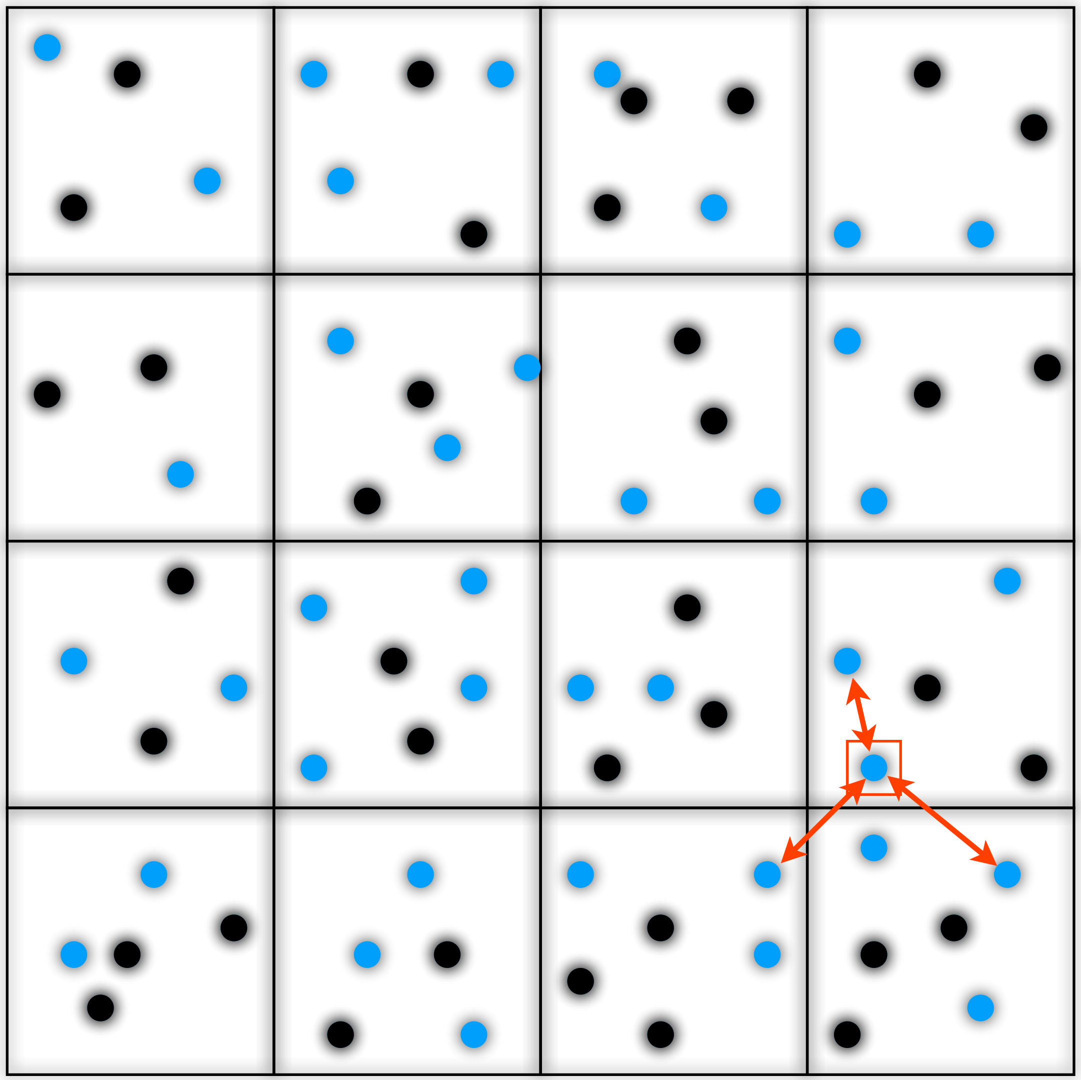

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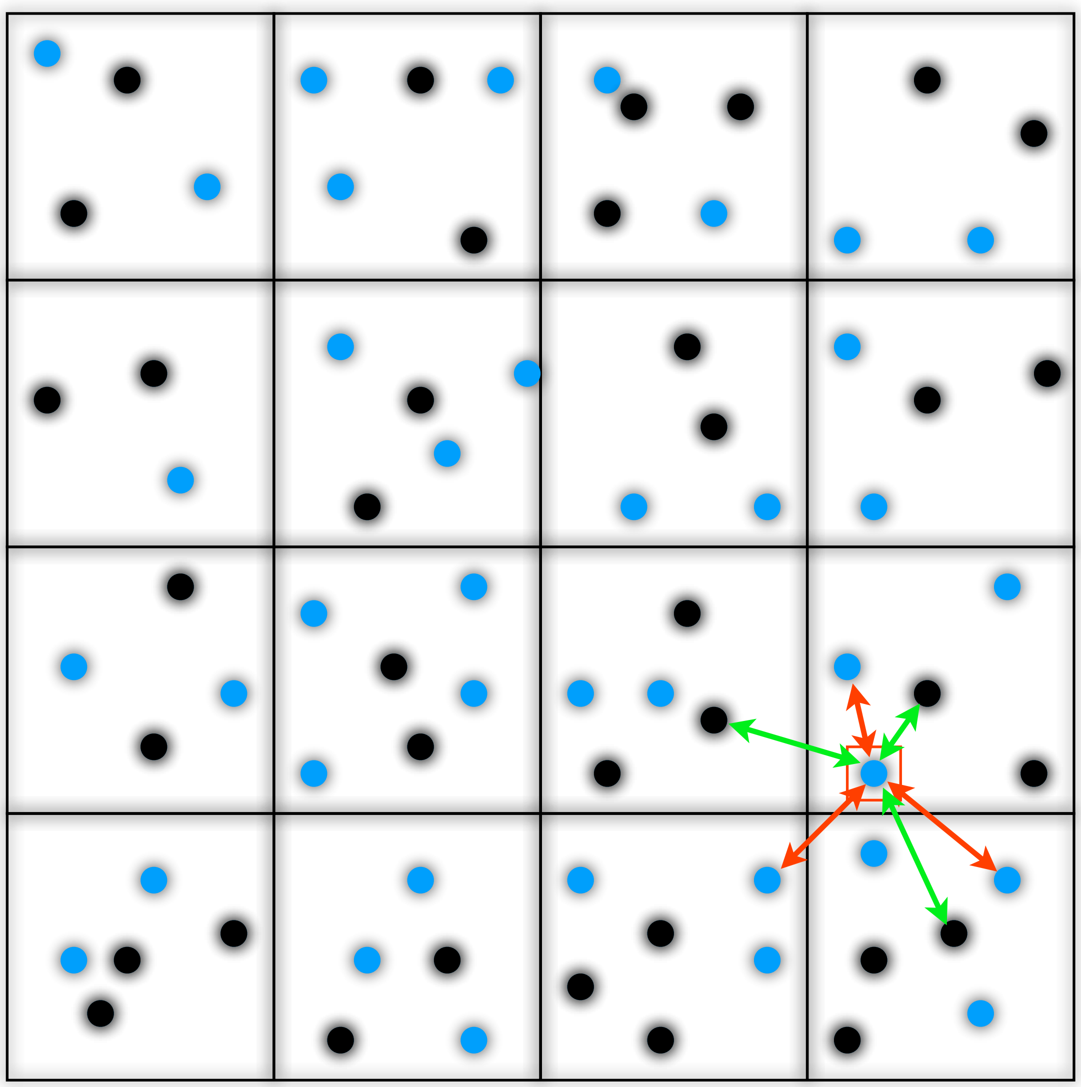

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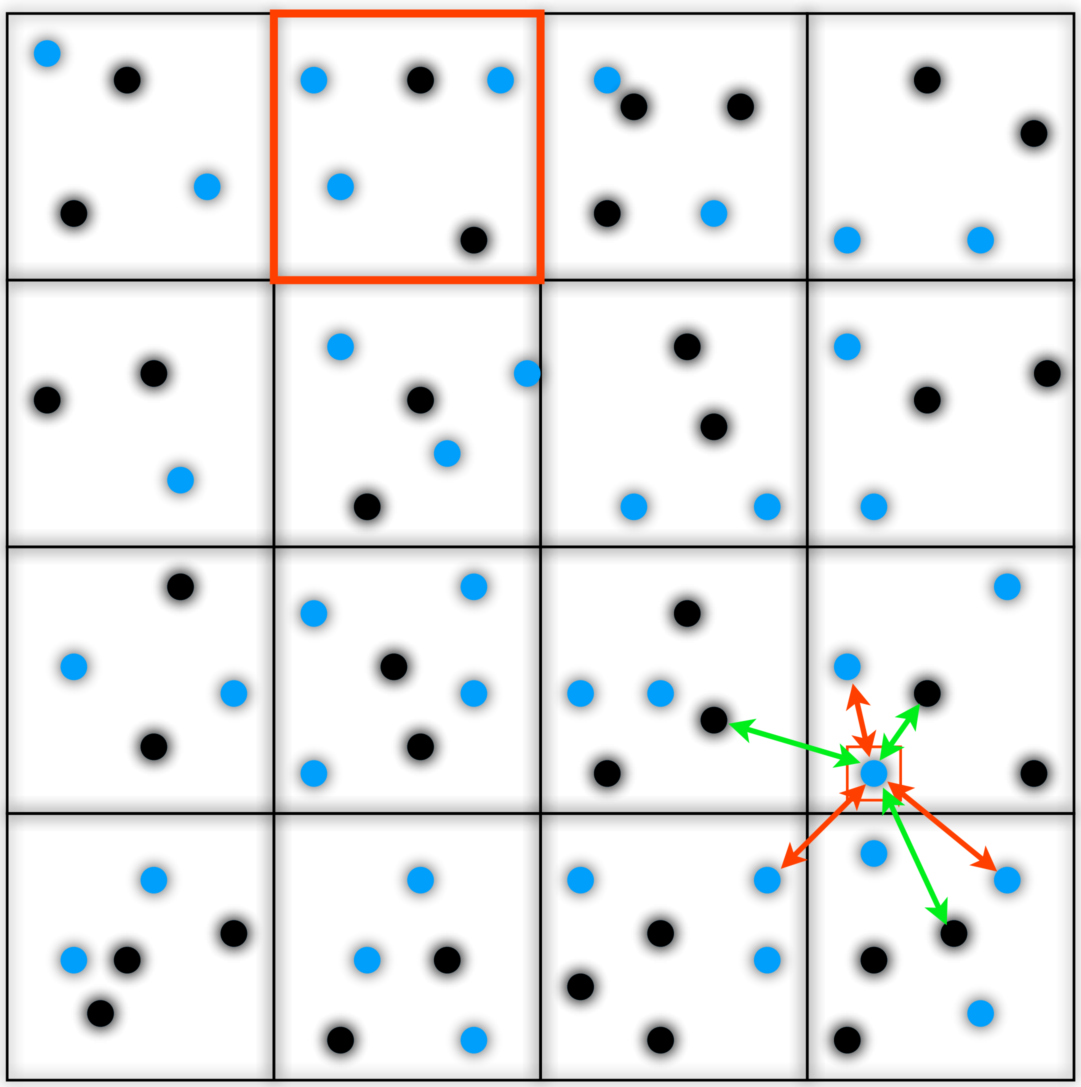

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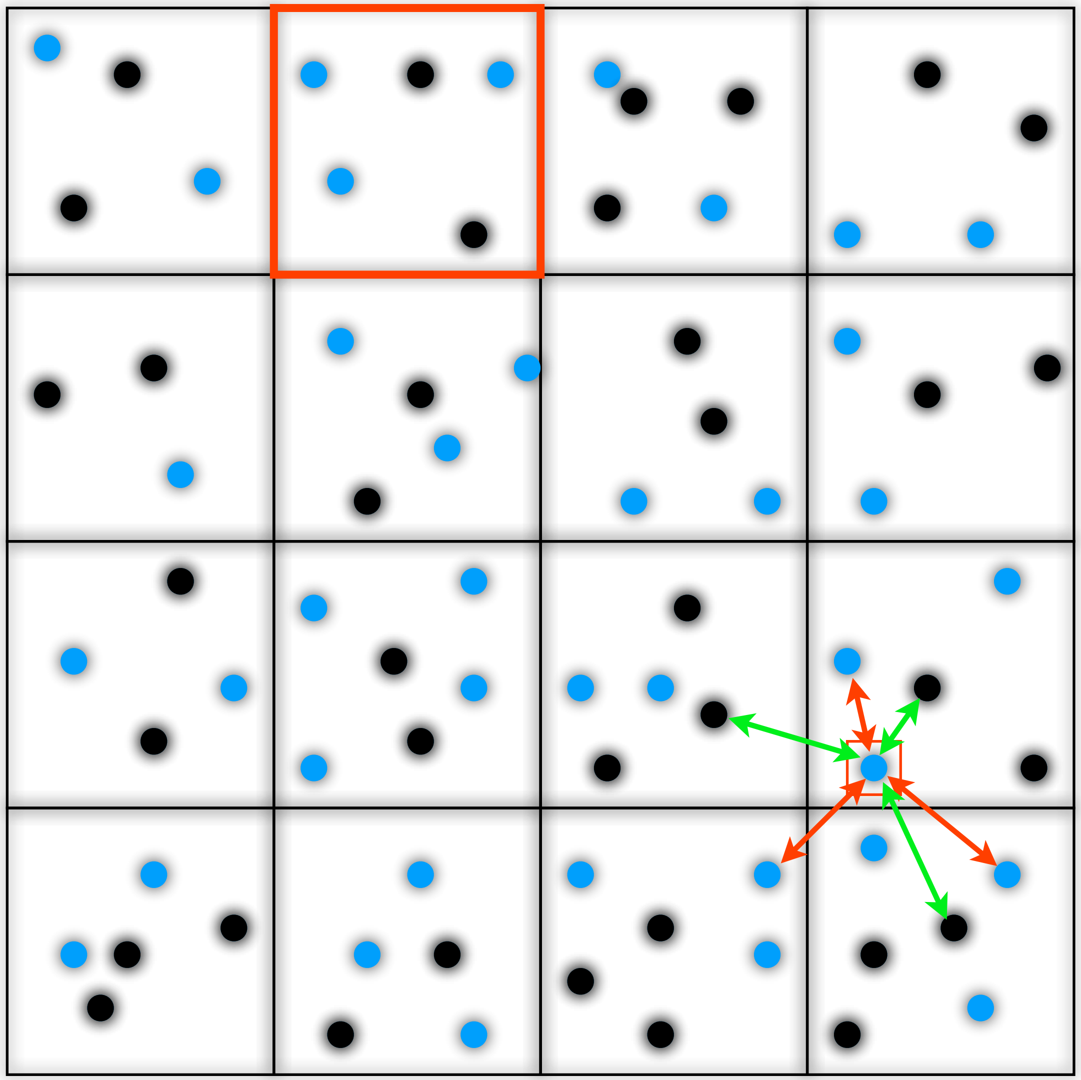

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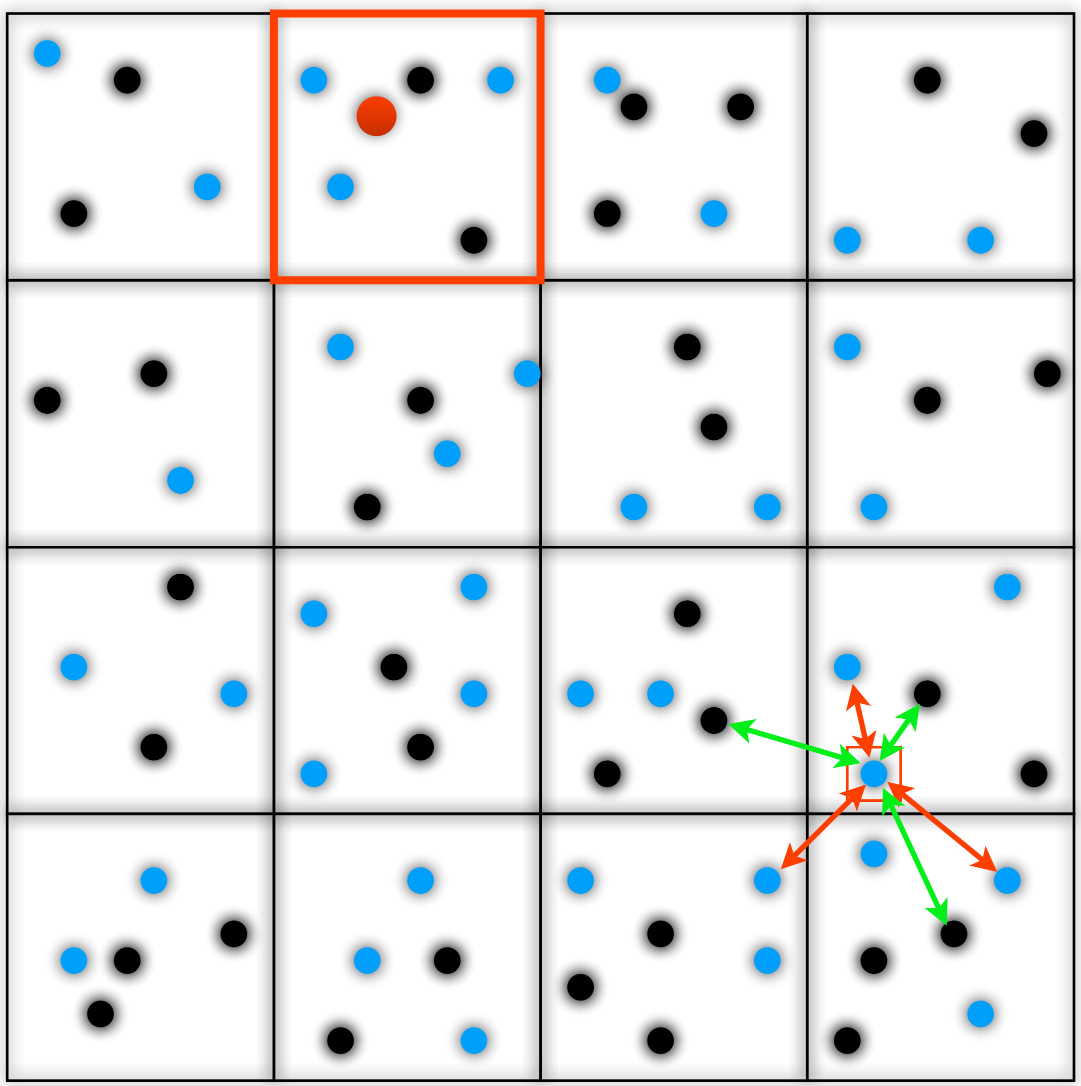

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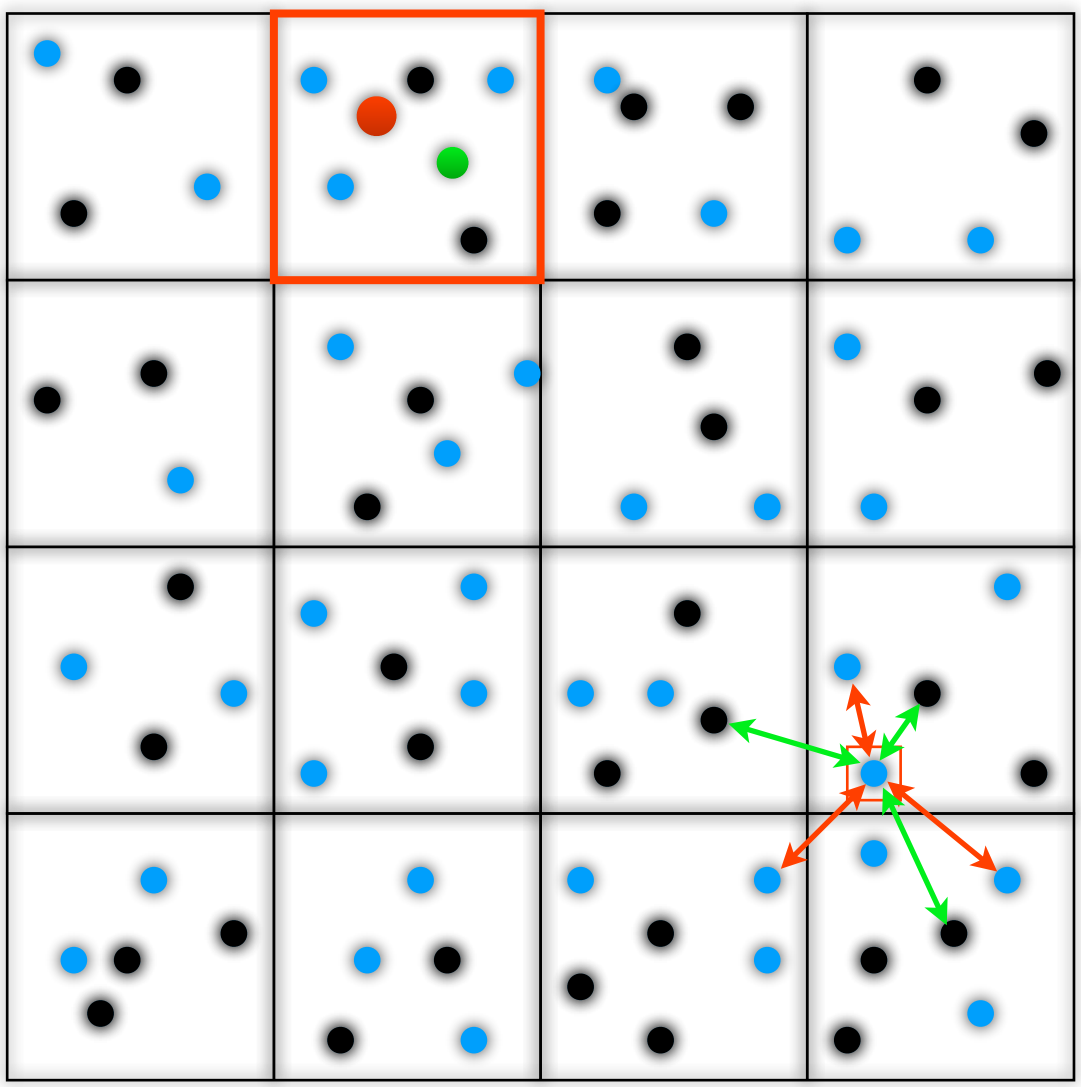

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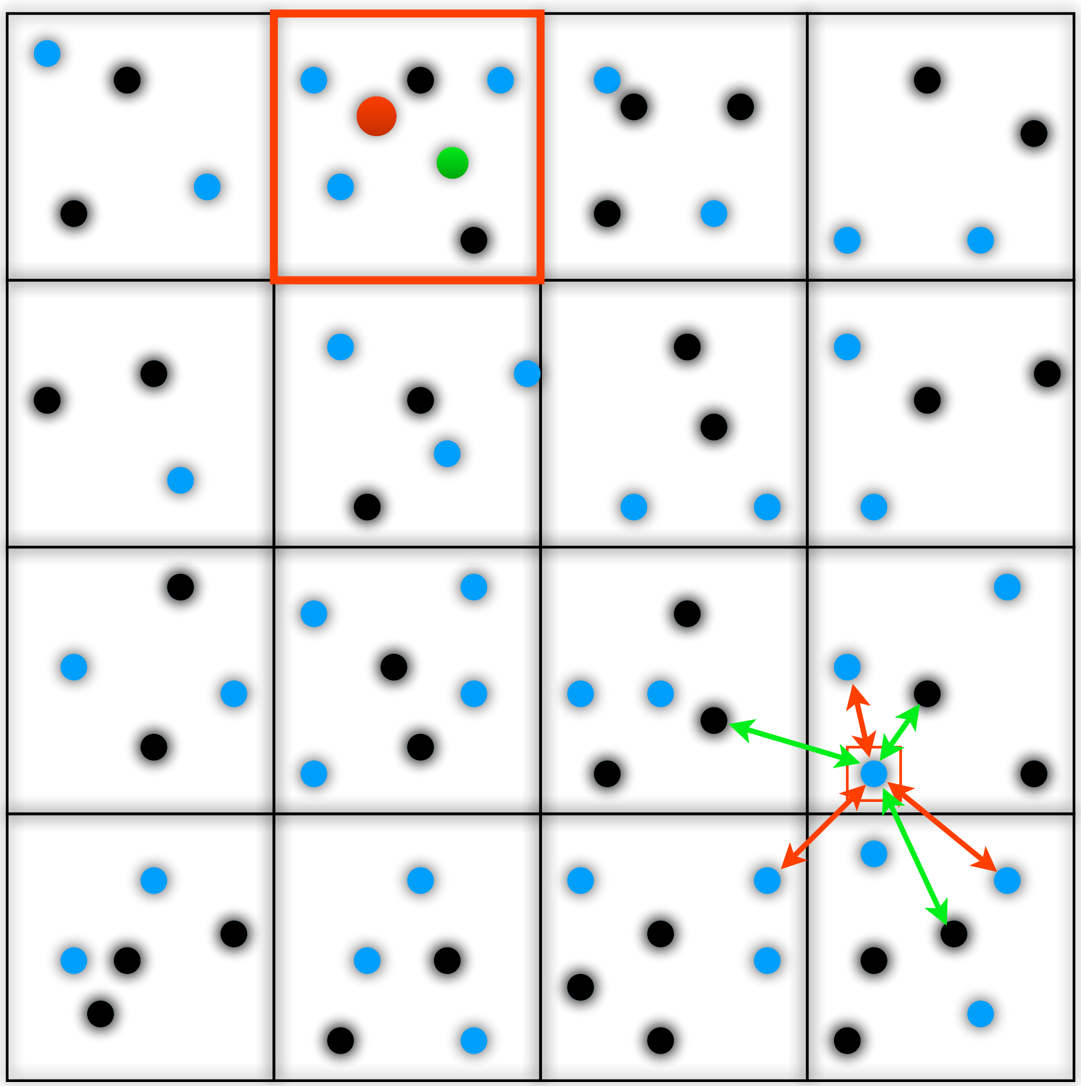
In interacting DE the coupling determines two different gravitational forces for dark matter and baryons:

Particle-Particle (Tree) for interacting DE



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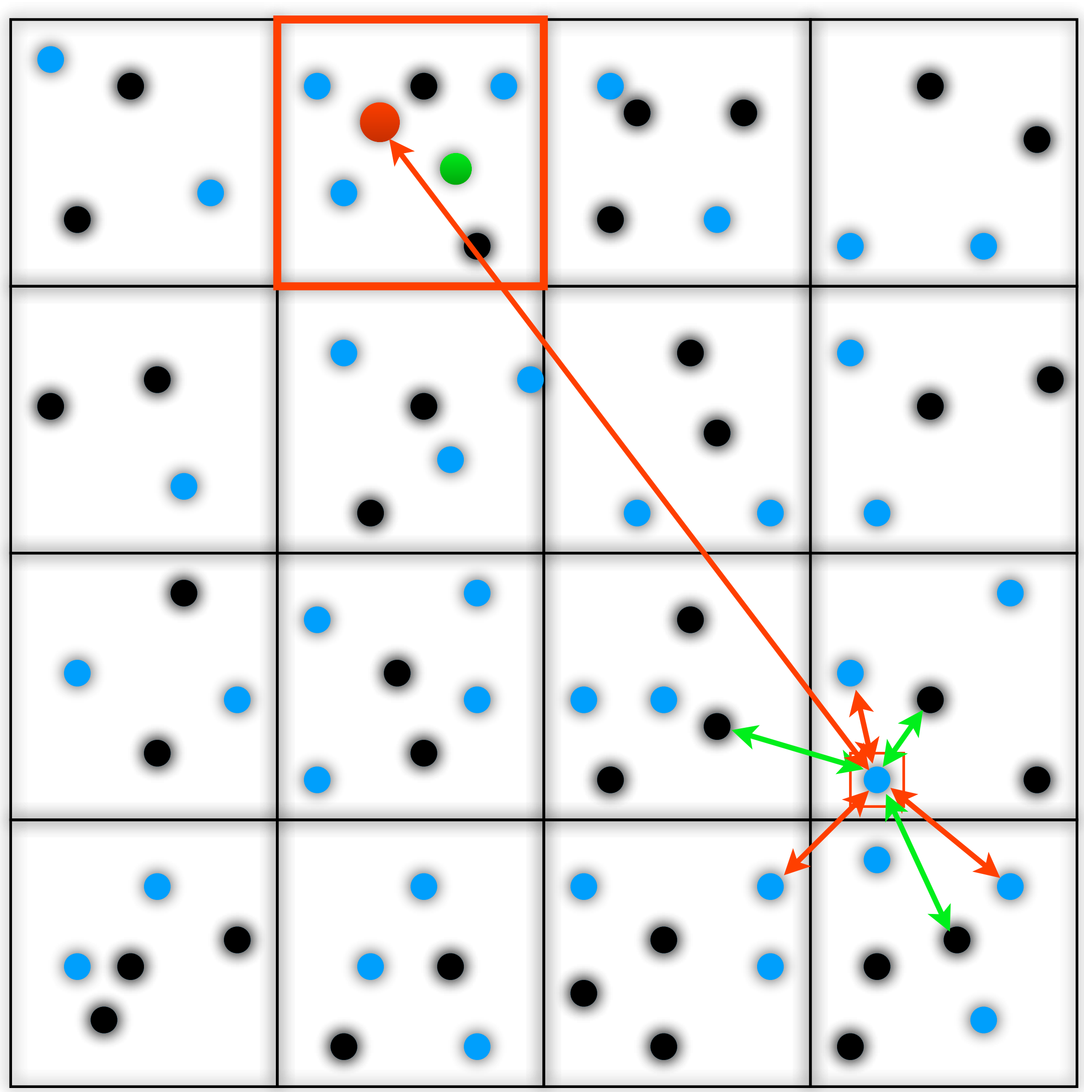

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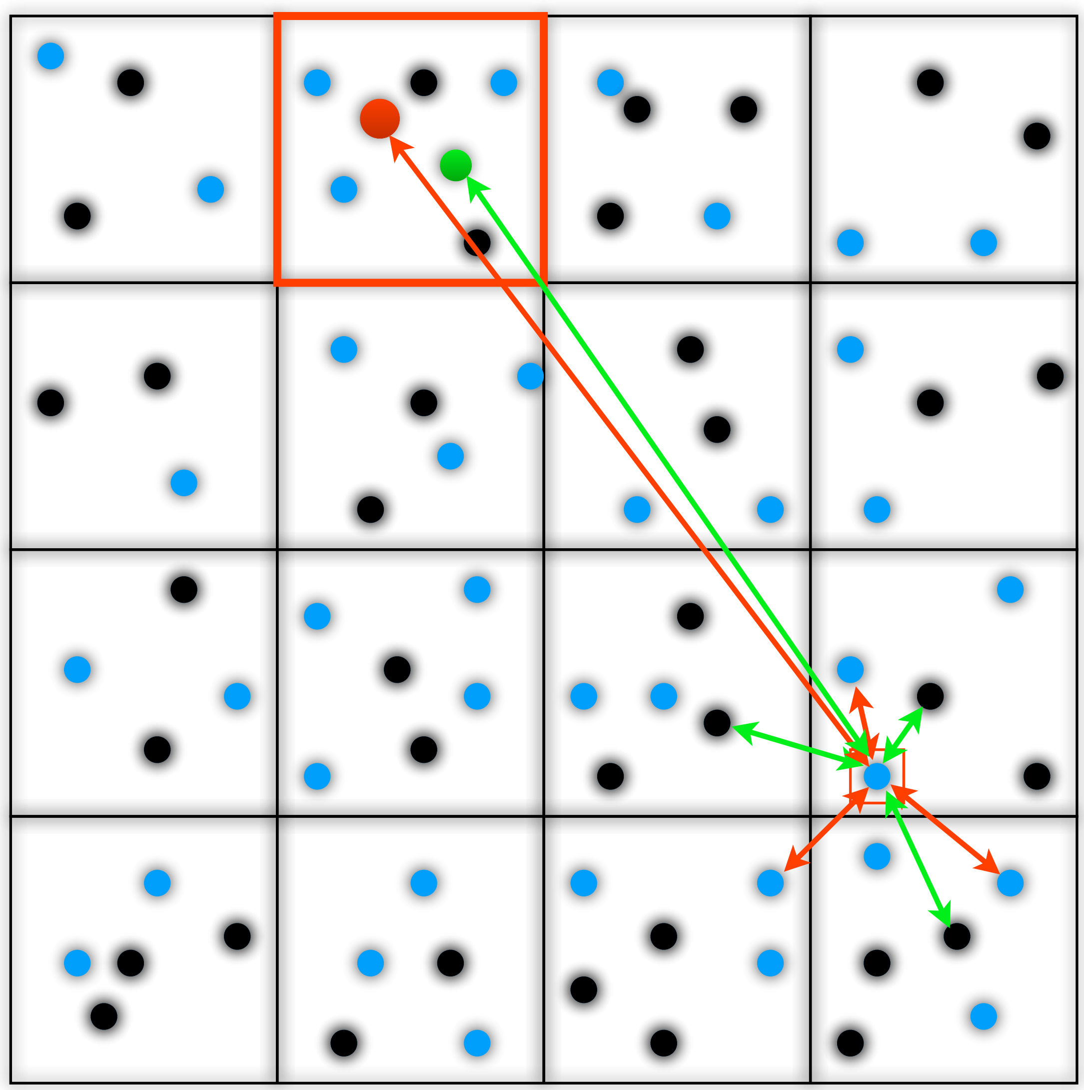

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$\delta \ll 1$

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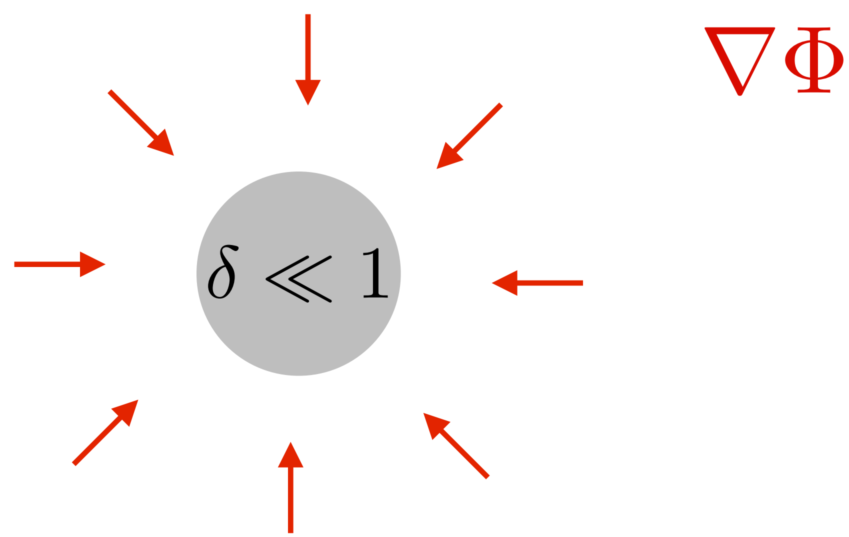

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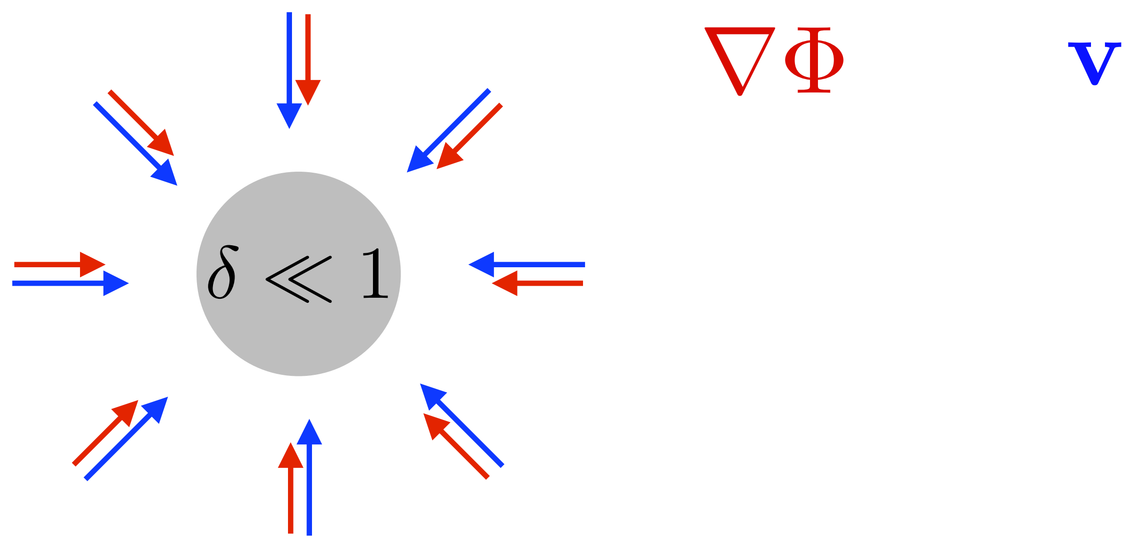

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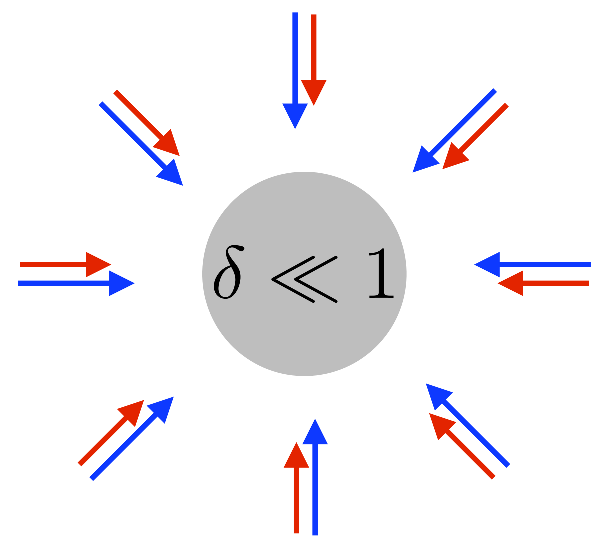

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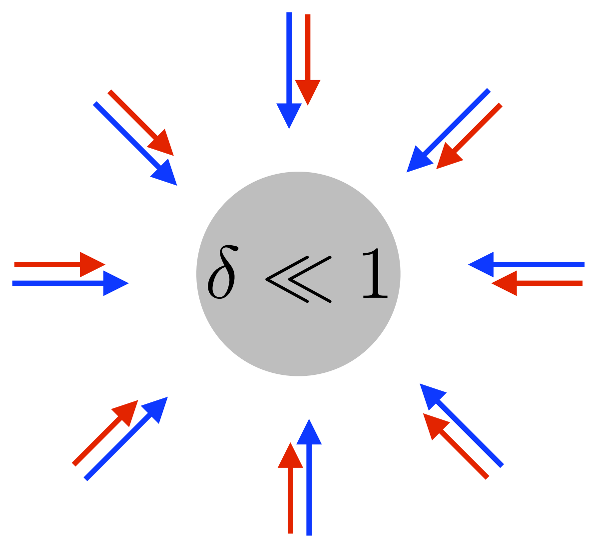

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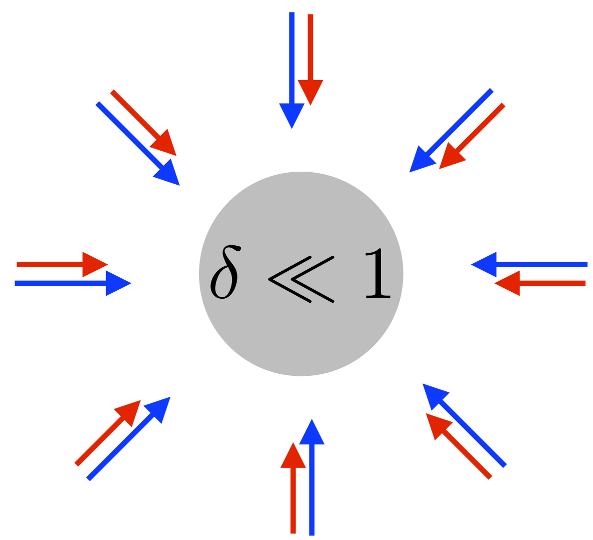

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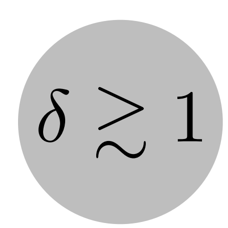

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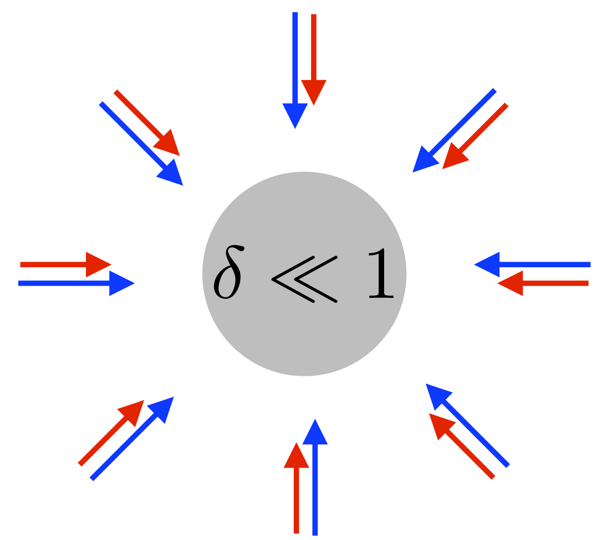

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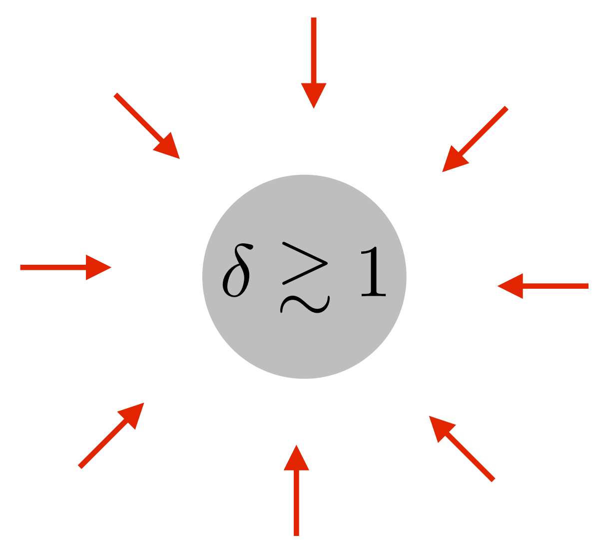

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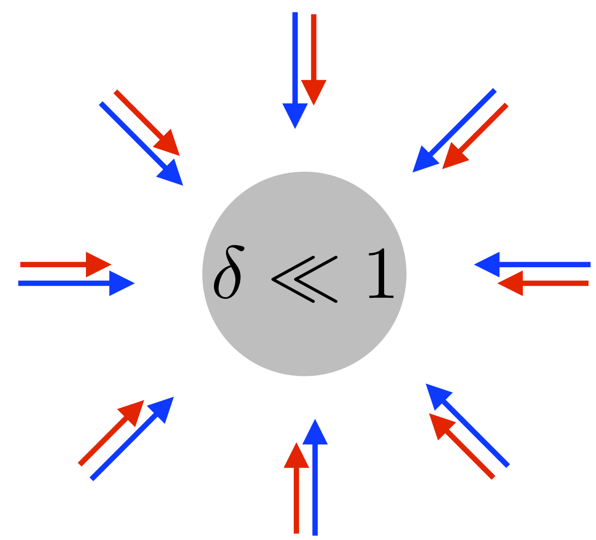

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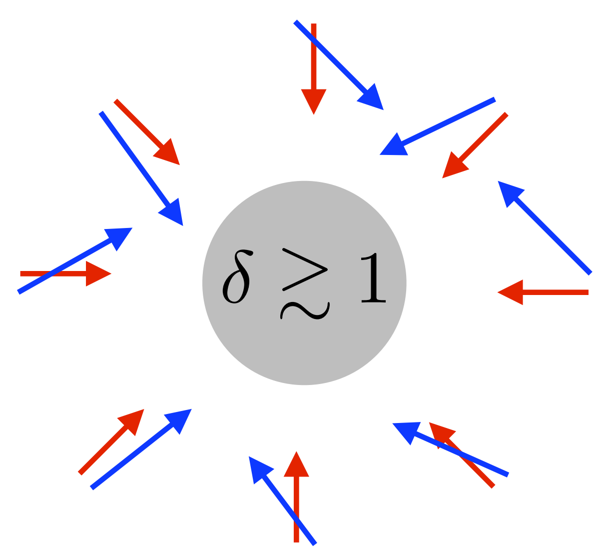

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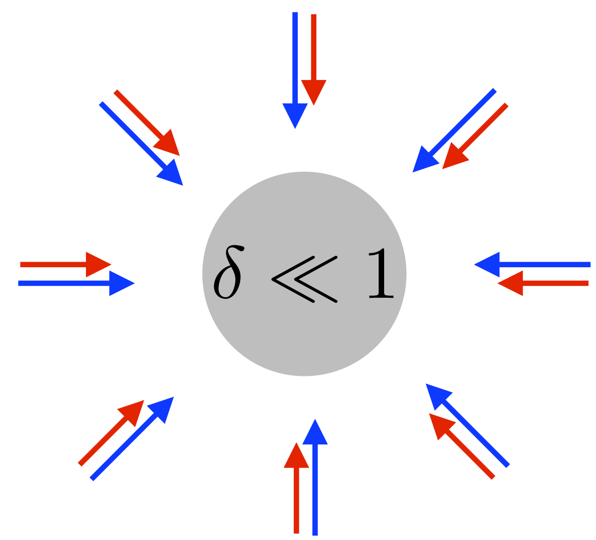

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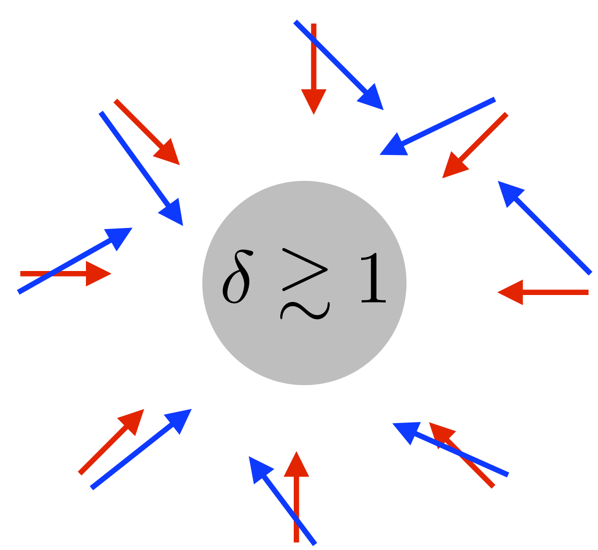

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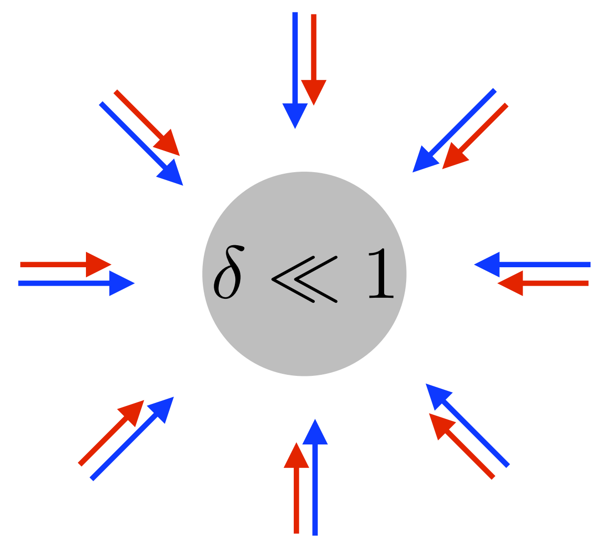

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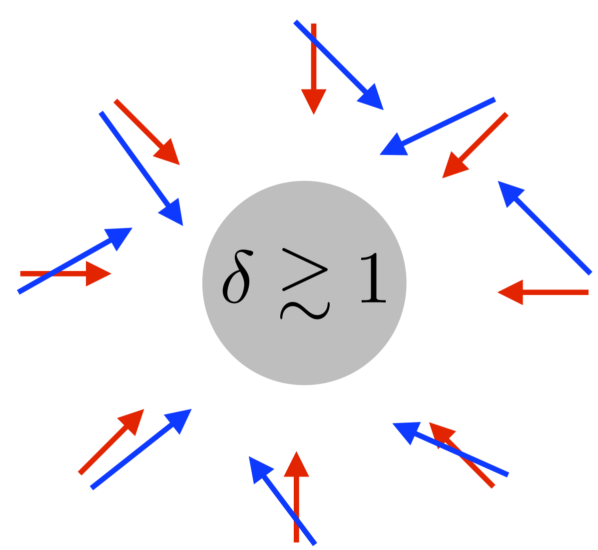

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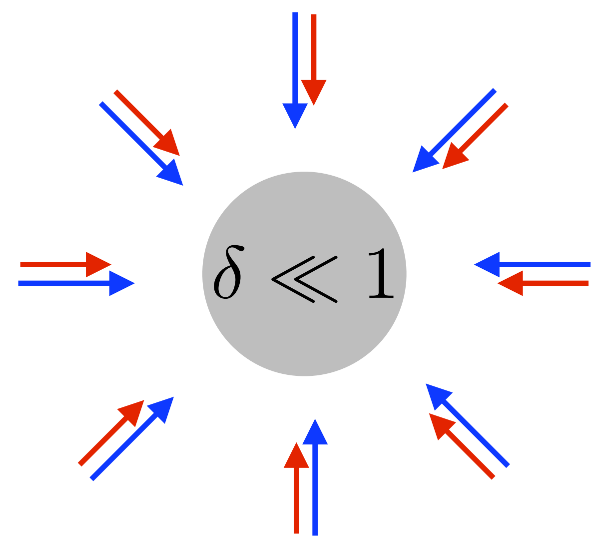

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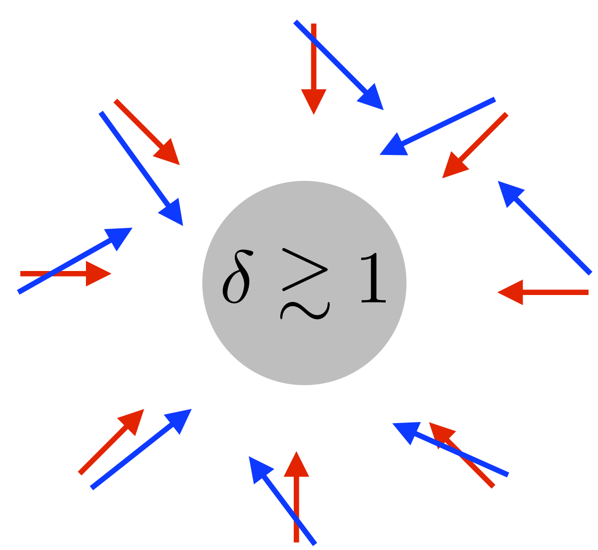

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Very different effects at linear and non-linear scales

Modified Gravity: f(R)

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An alternative approach to the Dark Energy problem is to modify General Relativity in the low curvature regime by changing the gravitational Action. One of the most popular models of this modified gravity approach is *f(R):*

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_m(g_{\mu\nu}, \Psi_m)$$

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By varying this Action with respect to the metric tensor, with a similar procedure as for the standard GR Action, one gets the *f(R)* field eqs:

$$f_R R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R = 8\pi G T_{\mu\nu}$$

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trace ⟶ $3\Box f_R + f_R R - 2f(R) = -8\pi G(\rho - 3p)$

where we have defined $f_R \equiv df/dR$

Modified Gravity: f(R)

In GR one has $f(R) = R - 2\Lambda$ so that $f_R = 1$ and $\Box f_R = 0$

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A particularly relevant model of *f(R)* gravity is given by the choice (Hu & Sawicki 2007):

$$f(R) = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \qquad m^2 \equiv \frac{8\pi G \rho_0}{3}$$

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In this setup the *f(R)* model will differ from the standard ΛCDM cosmology only at the level of linear and non-linear perturbations

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By perturbing this equation at linear order (posing $f_R = \bar{f}_R + \delta f_R$):

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta f_R = -M^2(f_R)\delta f_R + \frac{8\pi G}{3\bar{f}_R}\delta\rho_{\rm M}$$

That in the quasi-static approximation becomes:

$$\nabla^2\delta f_R = M^2(f_R)\delta f_R - \frac{8\pi G}{3\bar{f}_R}\delta\rho_{\rm M}$$

that is the same equation we saw for interacting dark energy.
However, differently from the case of interacting Dark Energy, in f(R) one CANNOT assume $M^2(f_R)\delta f_R \ll \delta\rho_{\rm M}$ **NOT TRUE**

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with the solution of the field configuration affecting structure formation through

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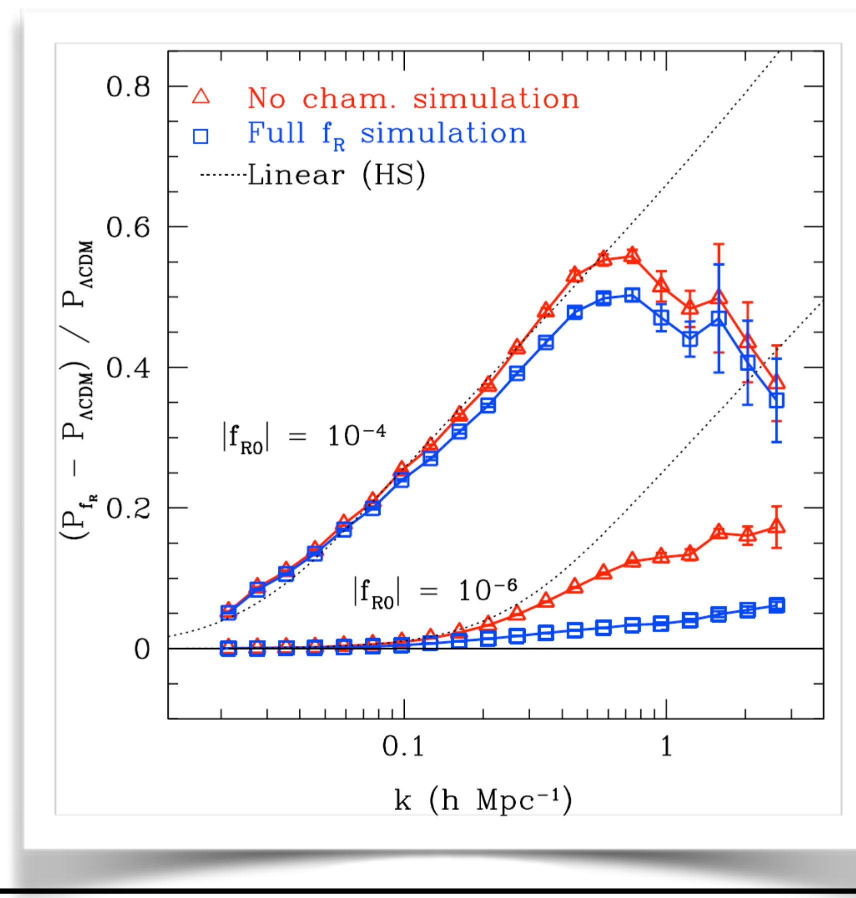
This has been implemented in several simulation codes after the first simulations performed by *Oyaizu et al. 2008*:



Ecosmog (Li et al. 2012)
MG-Gadget (Puchwein et al. 2013)
Isis (Llinares et al. 2014)
MG-Arepo (Arnold et al. 2019)

N-body algorithms for MG

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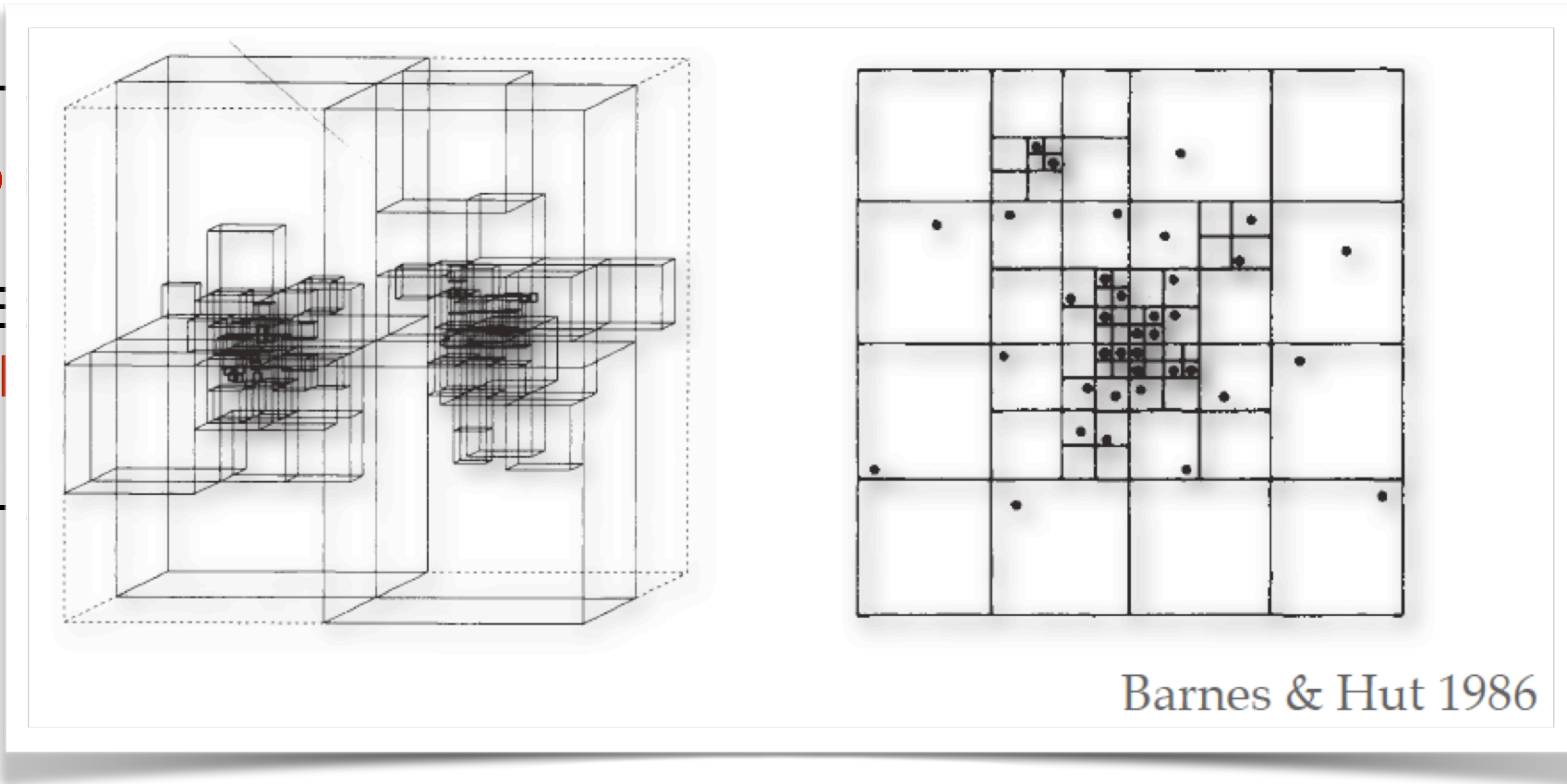

Barnes & Hut 1986

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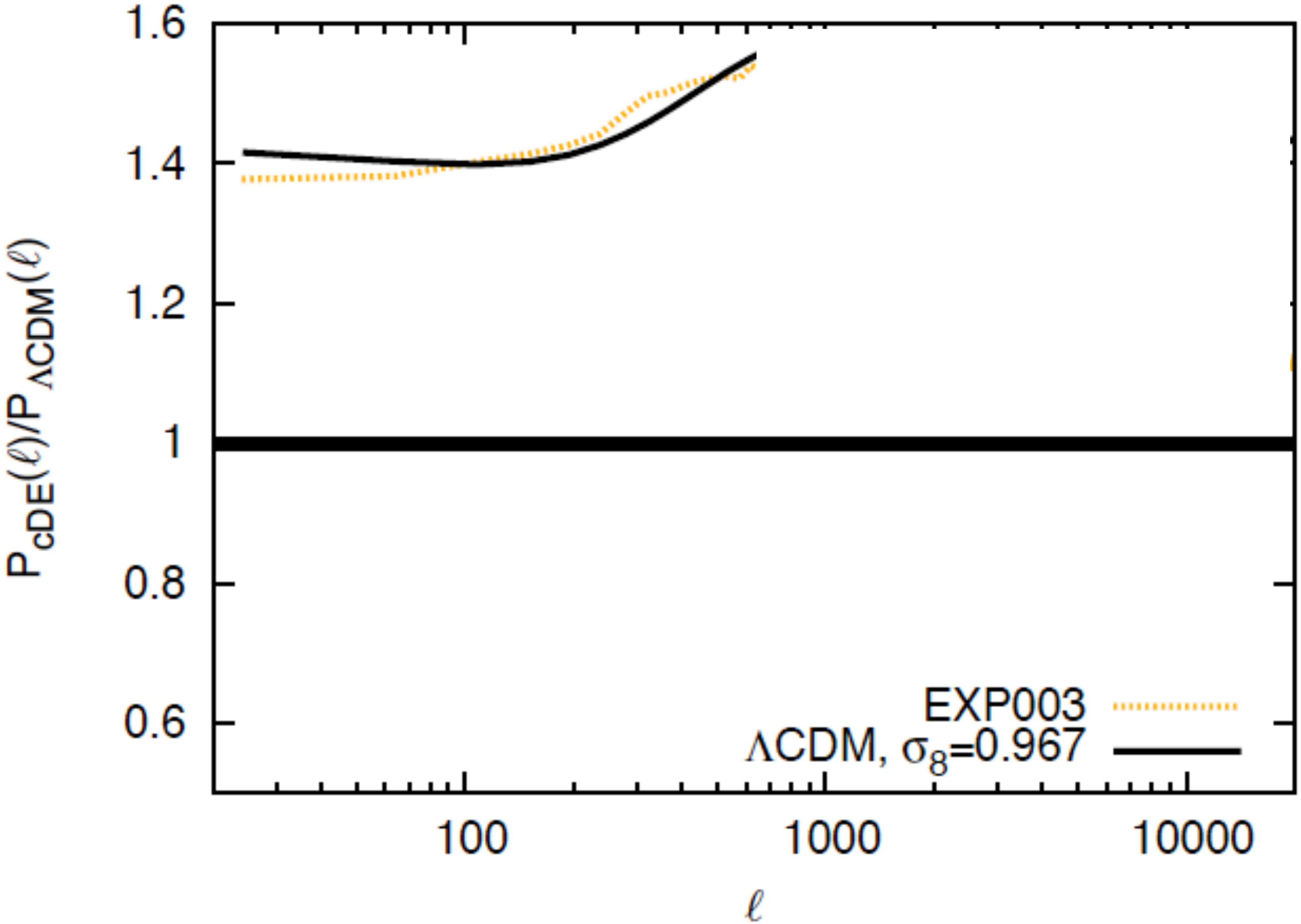
- Once f_R is known, $\delta R(f_R)$ is also known, and the Poisson equation can be solved by adding up the standard and the MG contributions

Why
doing Cosmological Simulations (for extended cosmological models)?

Interacting Dark Energy

Interacting Dark Energy

Lensing power spectrum extracted from N-body simulations with a ray-tracing technique (Pace, MB, et al. 2014)



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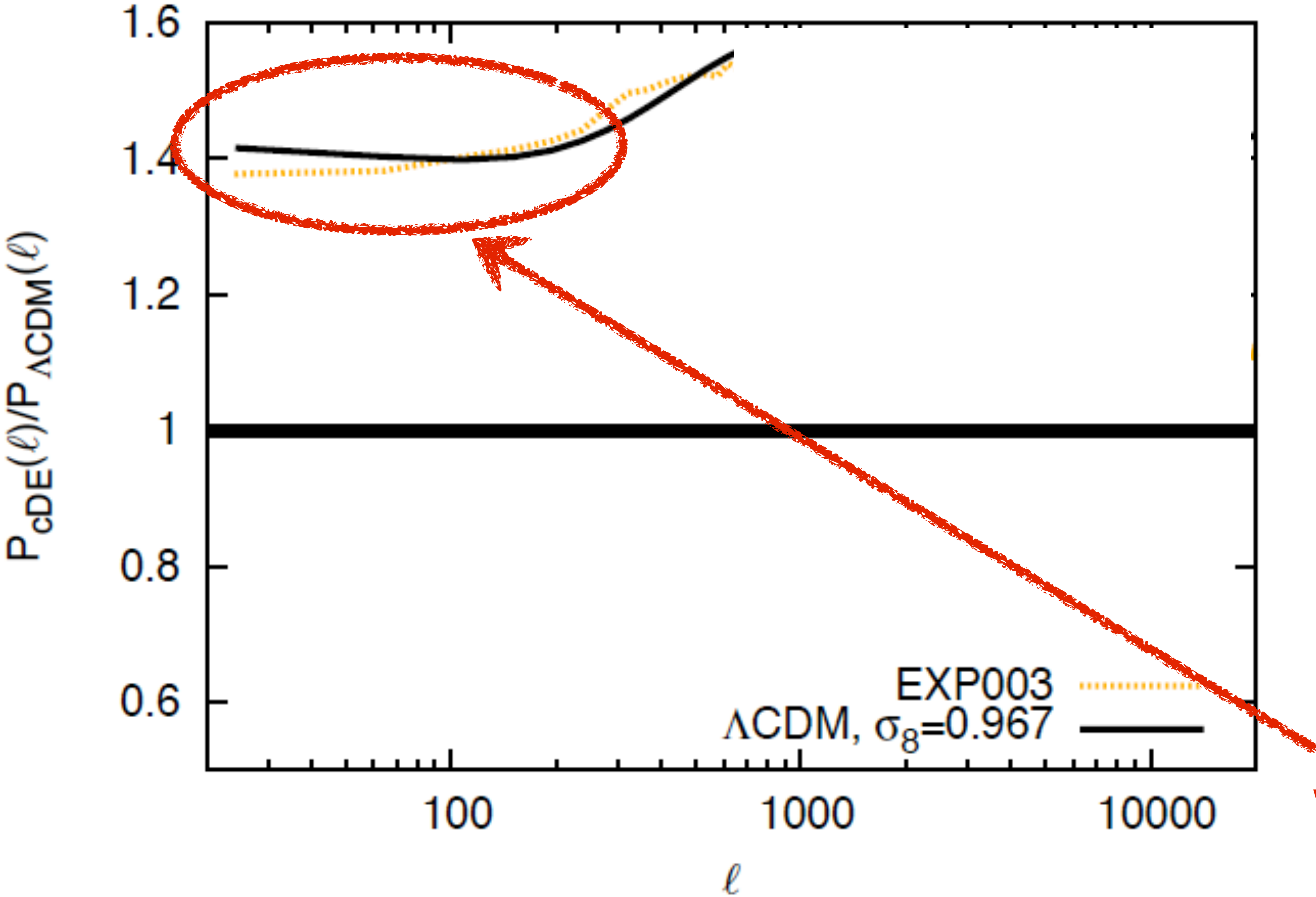

degeneracy with standard ΛCDM parameters

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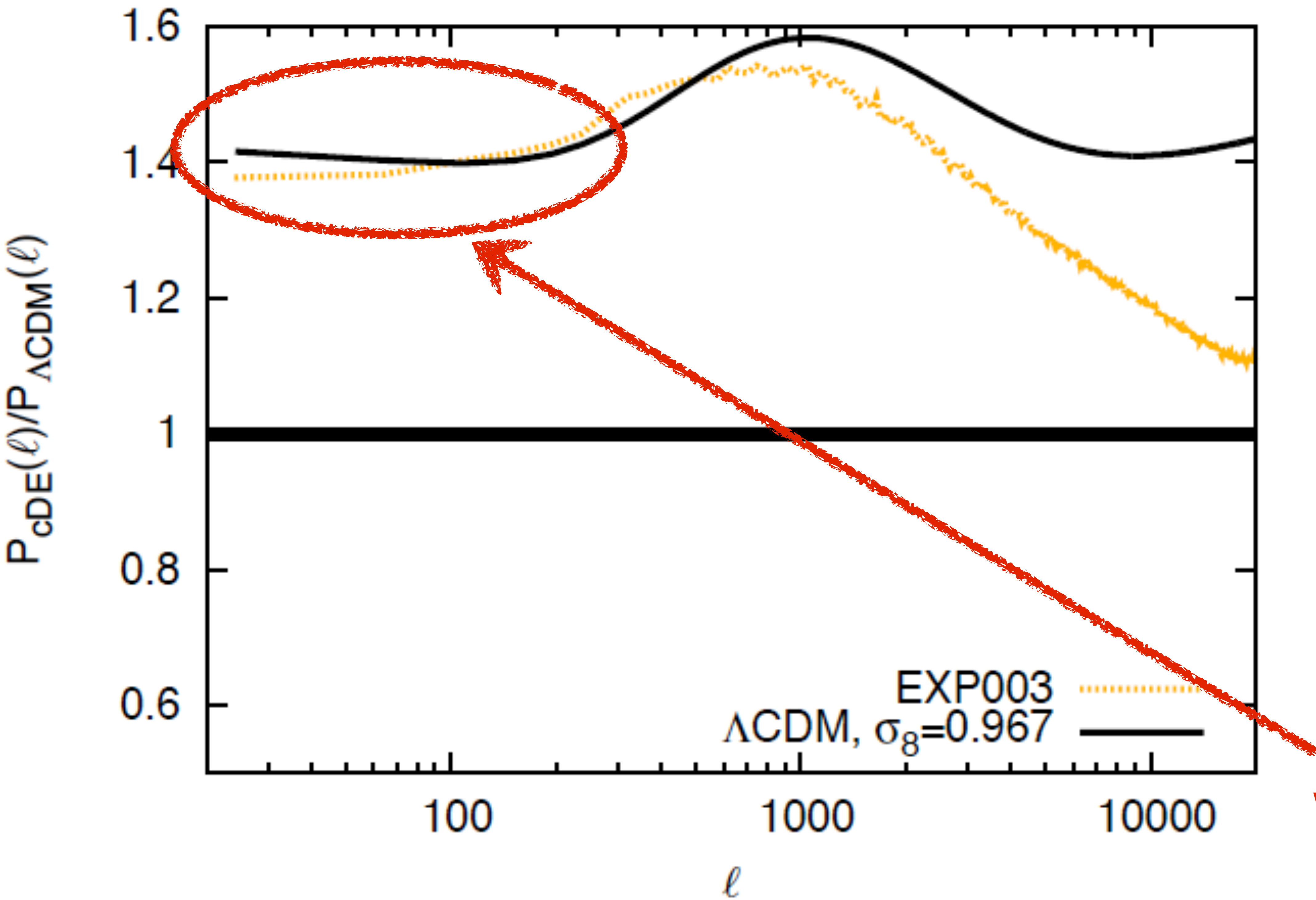

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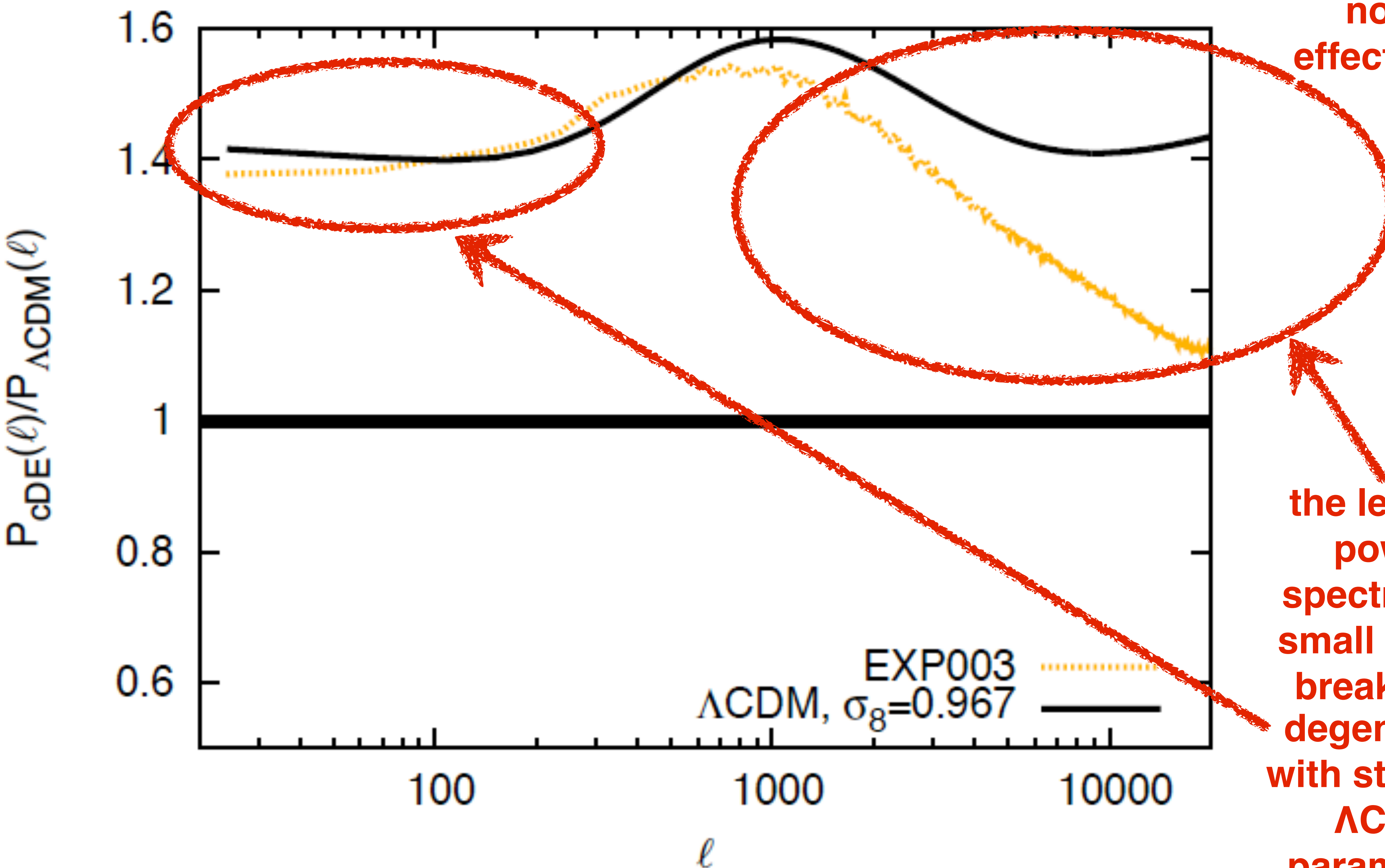

non-linear effect of friction term

the lensing power spectrum at small scales breaks the degeneracy with standard ΛCDM parameters

The effect of Massive Neutrinos

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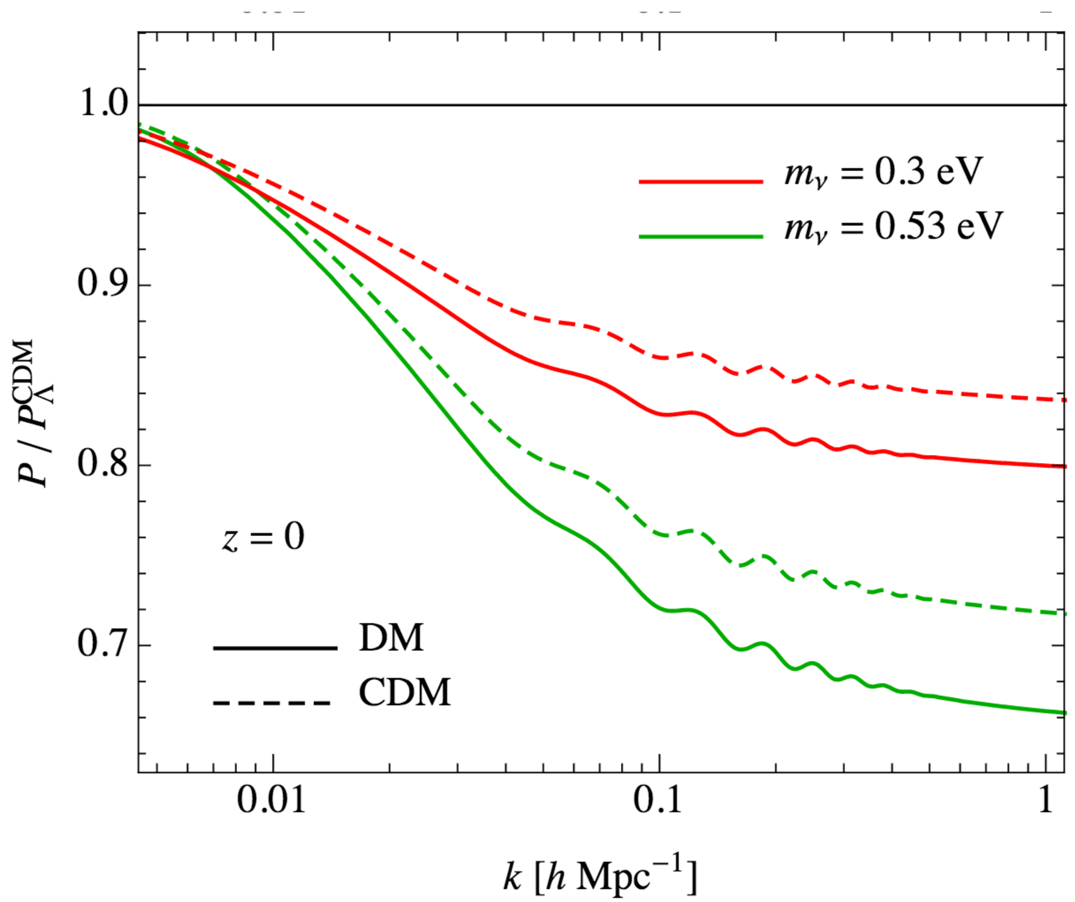
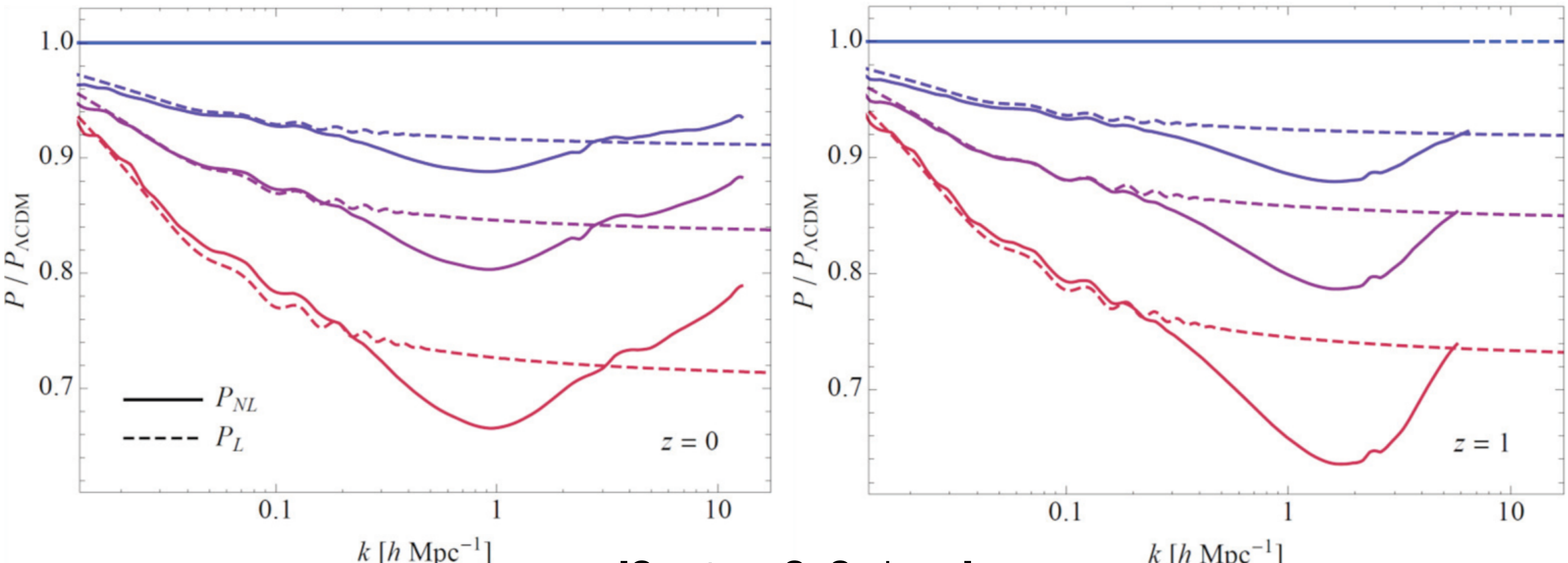


Free streaming suppresses structure at small scales

Castorina et al. 2016

The effect of Massive Neutrinos

m_ν = 0.17; 0.3; 0.53 eV



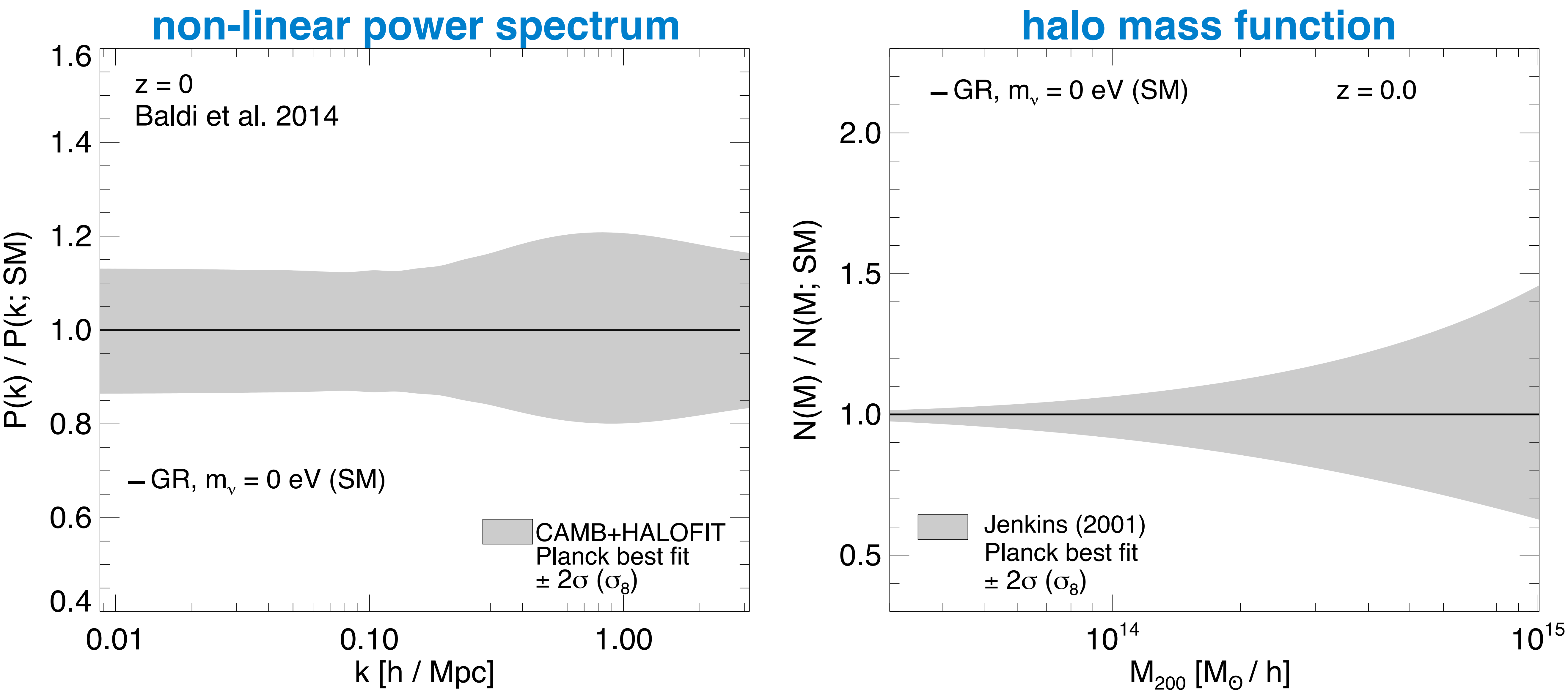
[Courtesy C. Carbone]

[see also Bird, Viel & Haehnelt 2012]

Nonlinear suppression of the matter P(k) ~15% larger than linear predictions at k~1-2 h/Mpc (critical range of scales for WL surveys)

The f(R)-massive neutrinos degeneracy

MB et al. 2014



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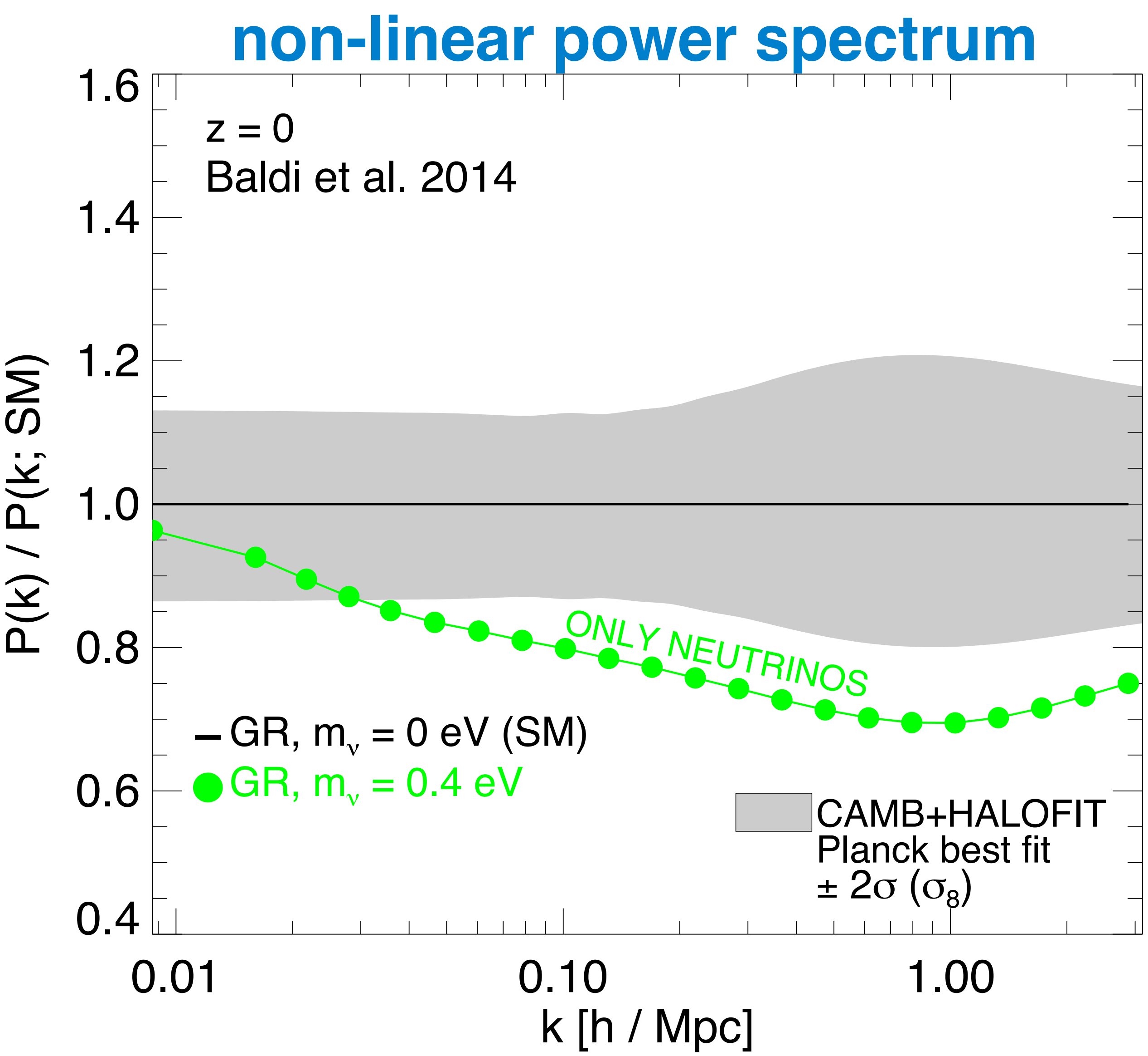
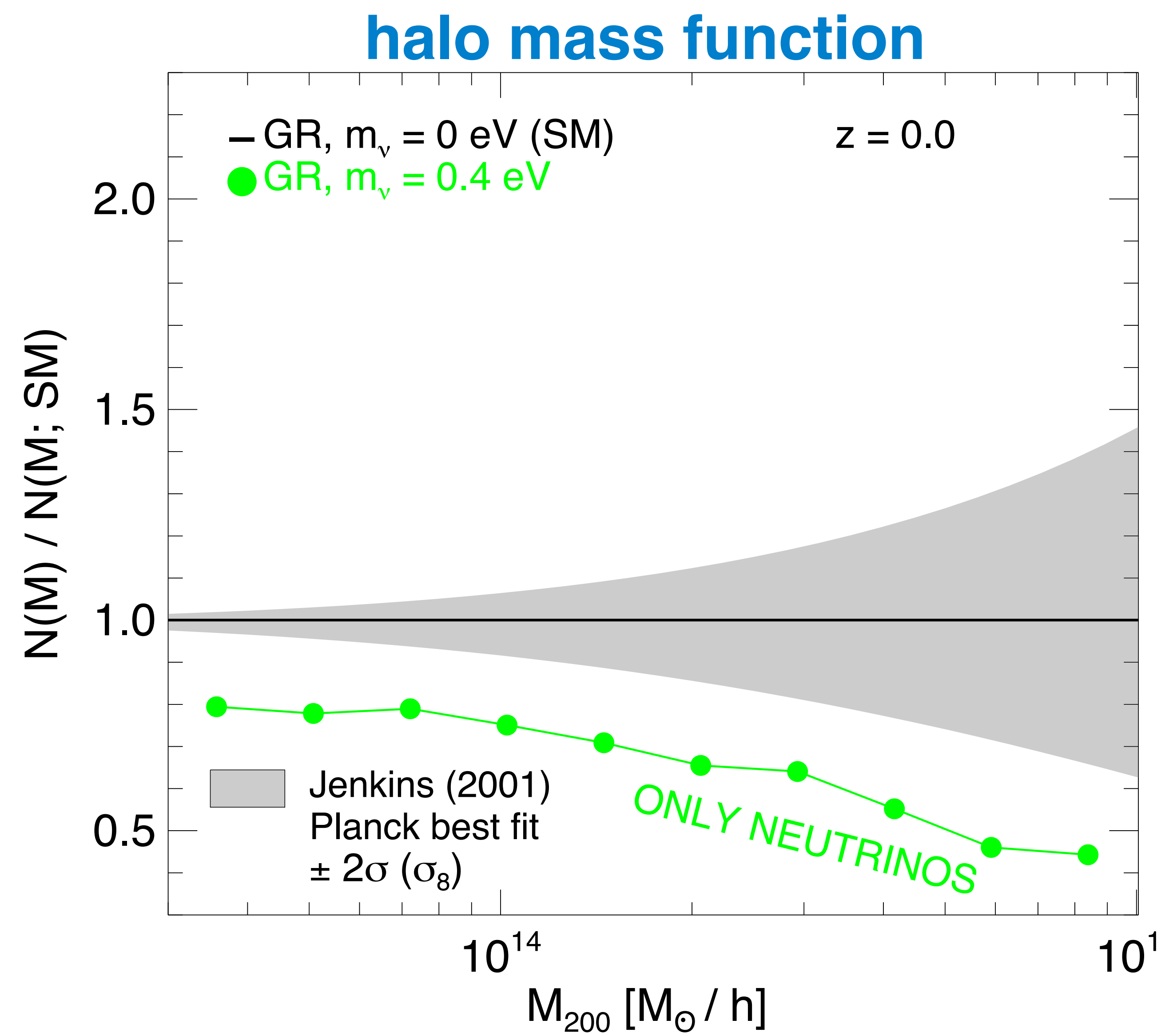
MB et al. 2014

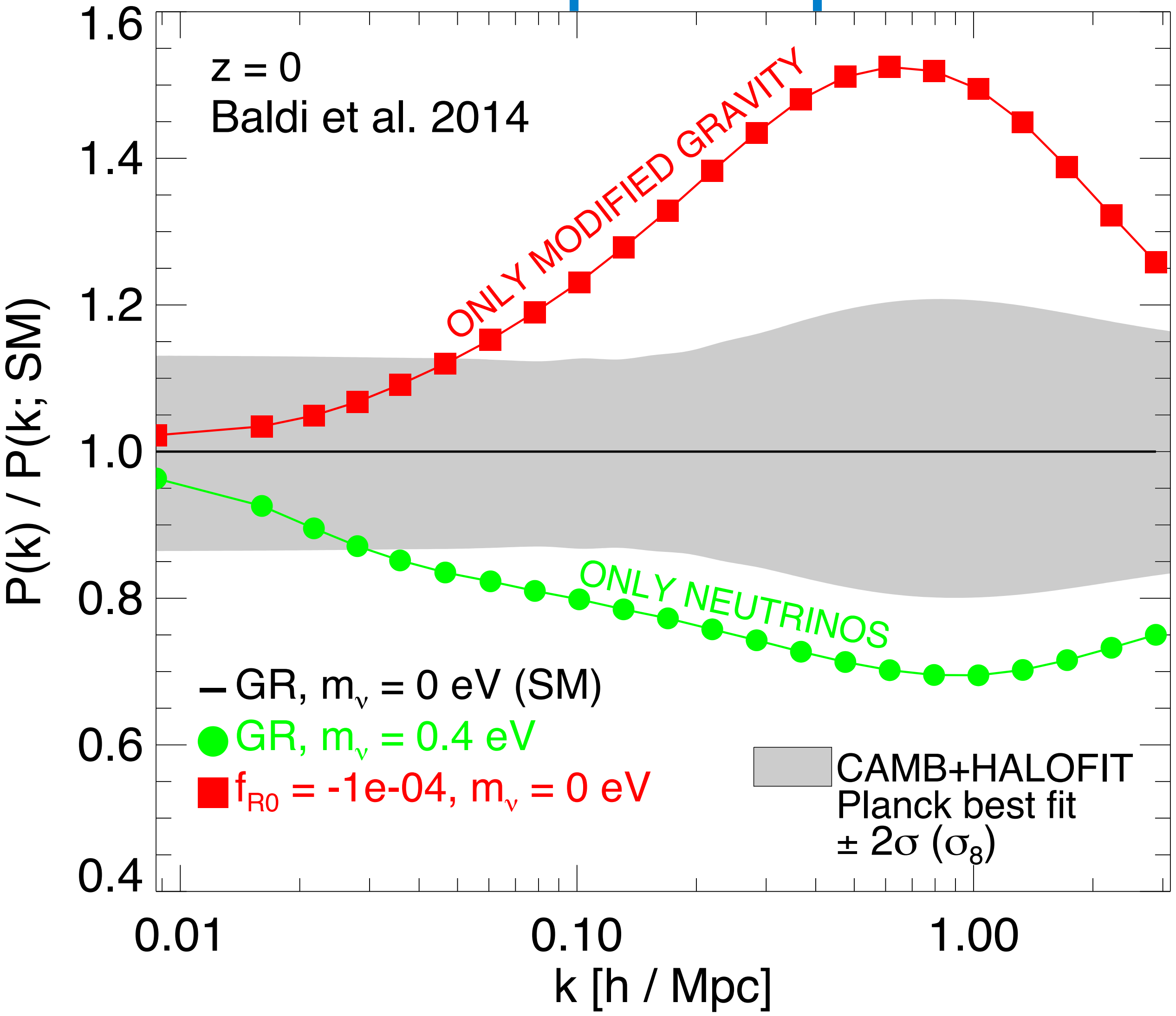




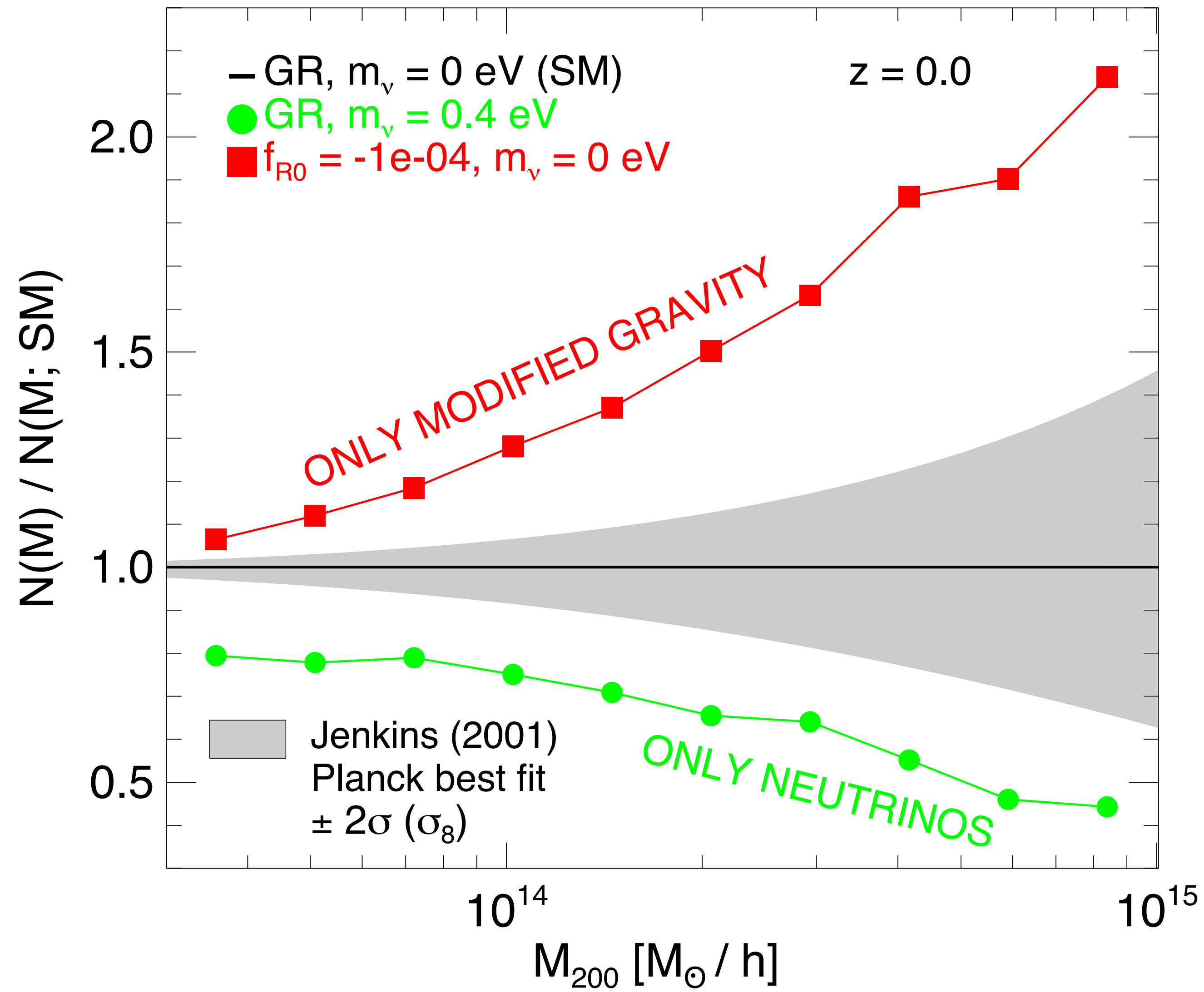
The f(R)-massive neutrinos degeneracy

MB et al. 2014

non-linear power spectrum

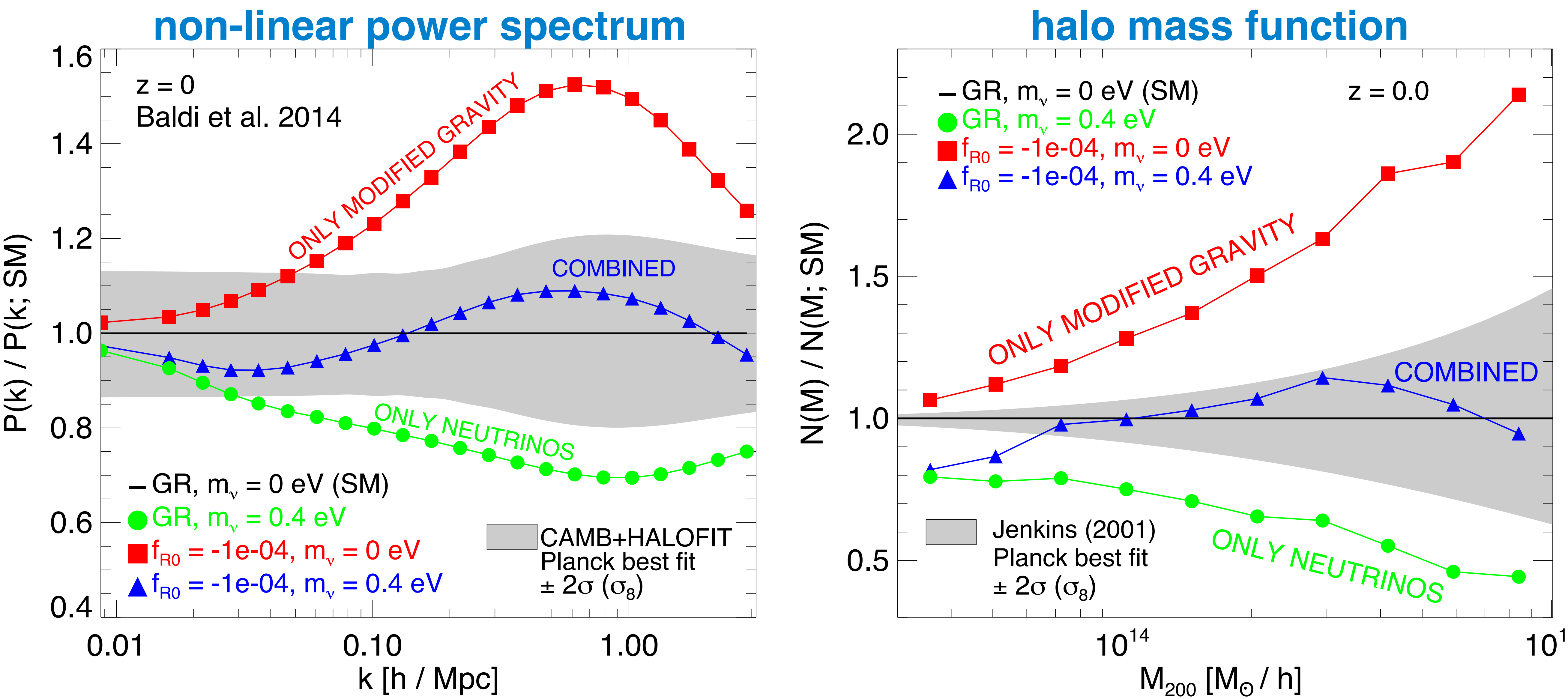


halo mass function



The f(R)-massive neutrinos degeneracy

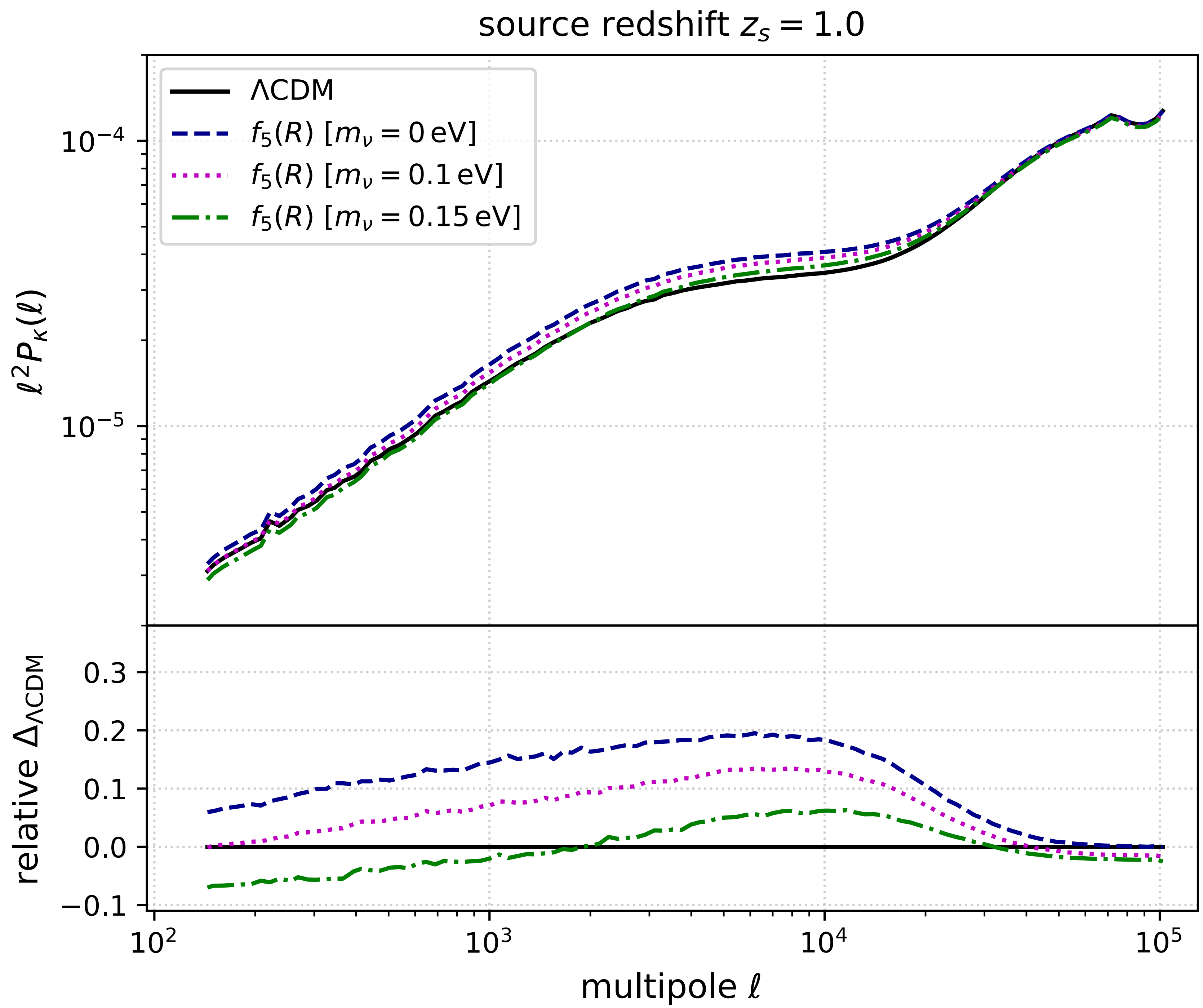
MB et al. 2014



The deviations expected for a non-zero neutrino mass and for an f(R) theory of gravity are suppressed below observational resolution if both phenomena coexist → **RISK OF MISINTERPRETING THE DATA!!!**

The f(R)-massive neutrinos degeneracy

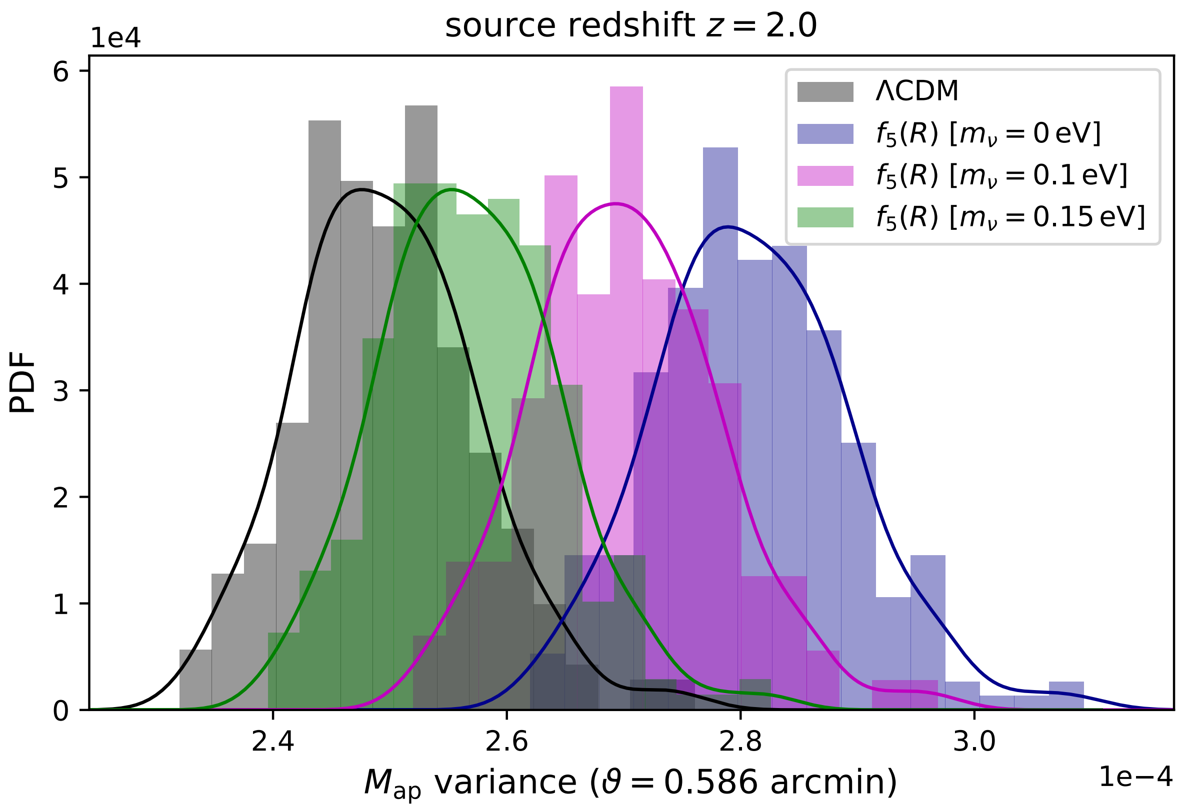
Peel, MB et al., A&A 2018 (arXiv:1805:05146)



The f(R)-massive neutrinos degeneracy

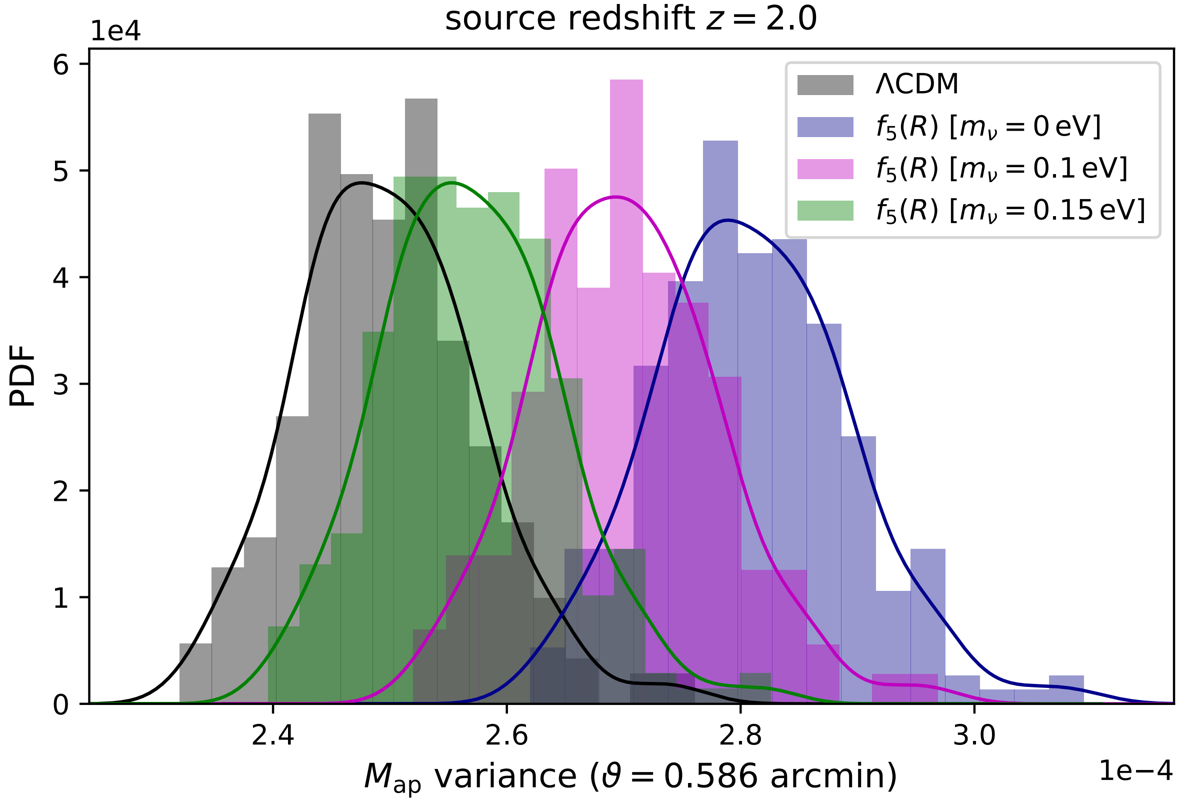
The f(R)-massive neutrinos degeneracy

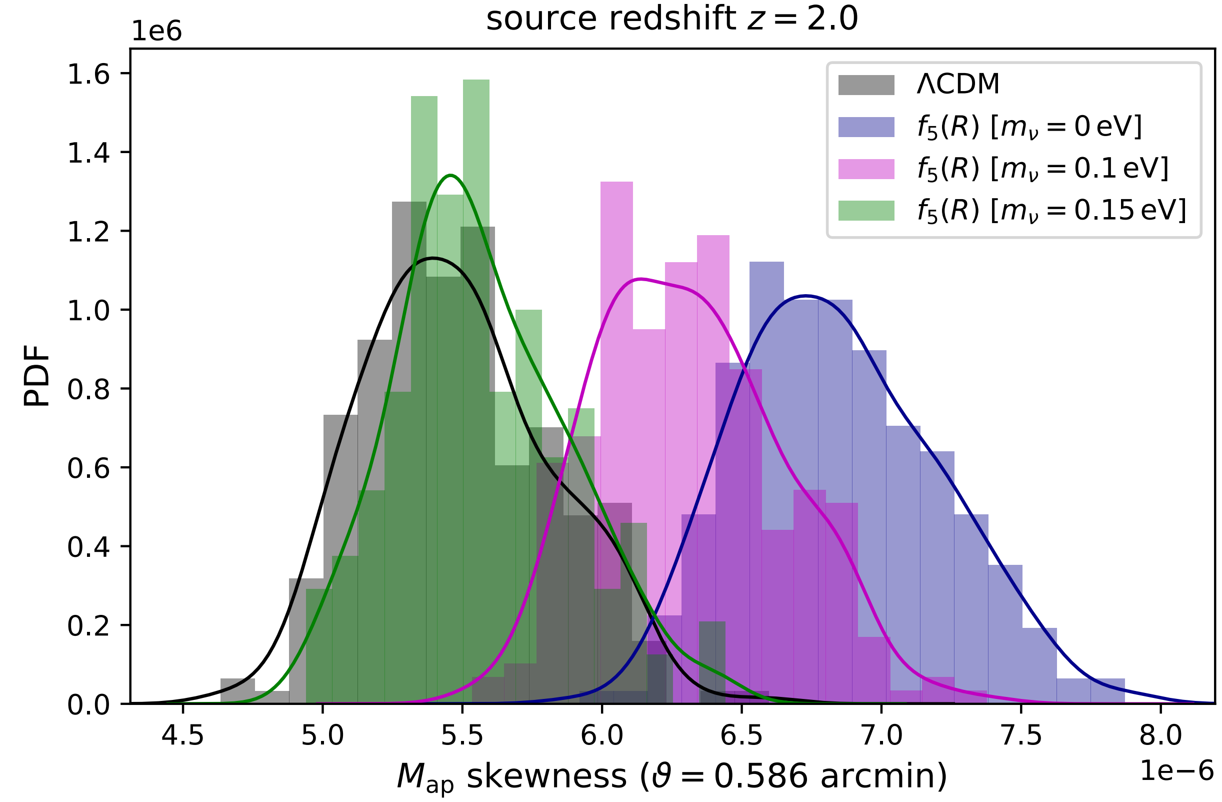
Peel et al. 2018



The f(R)-massive neutrinos degeneracy

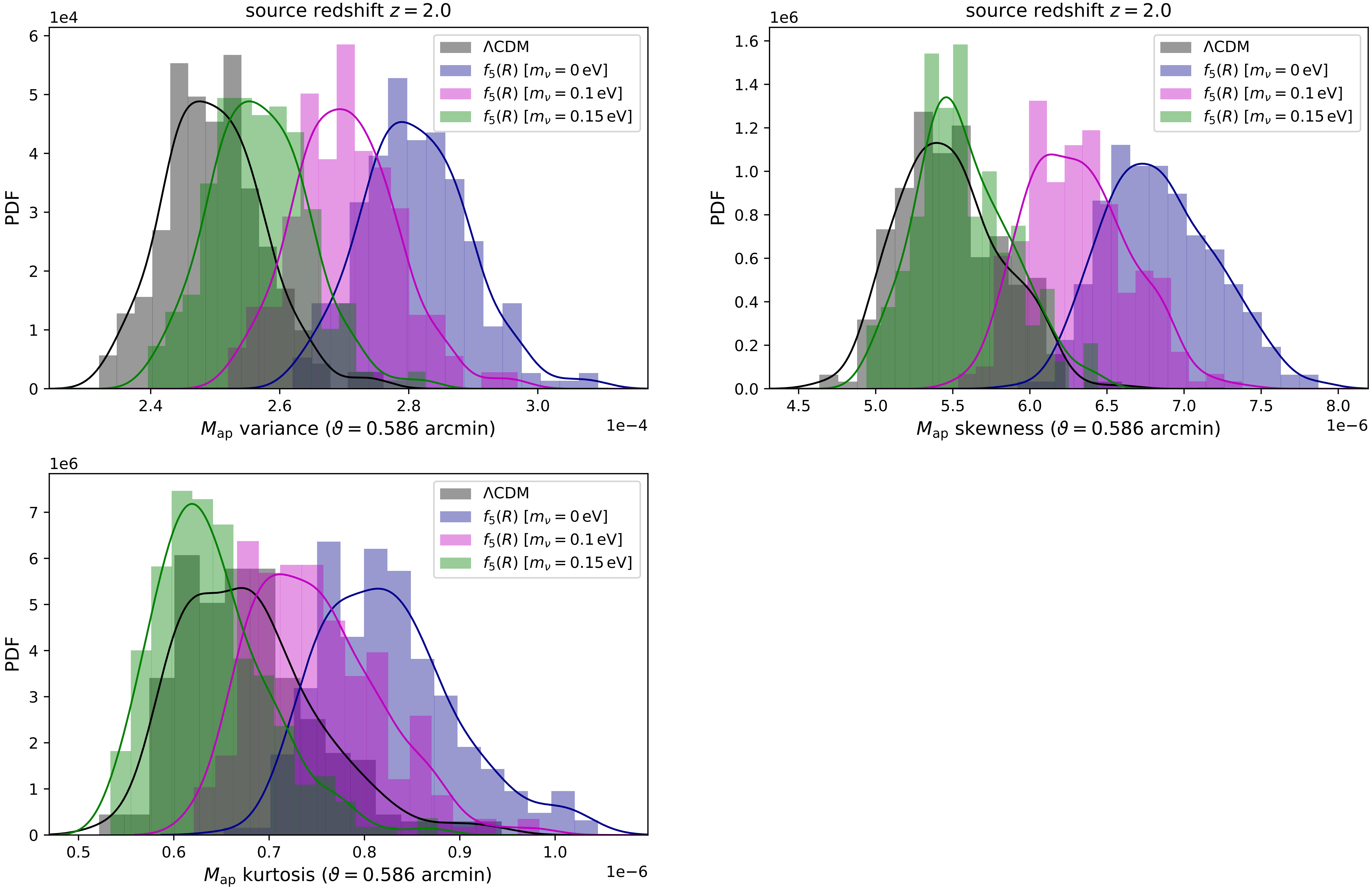
Peel et al. 2018





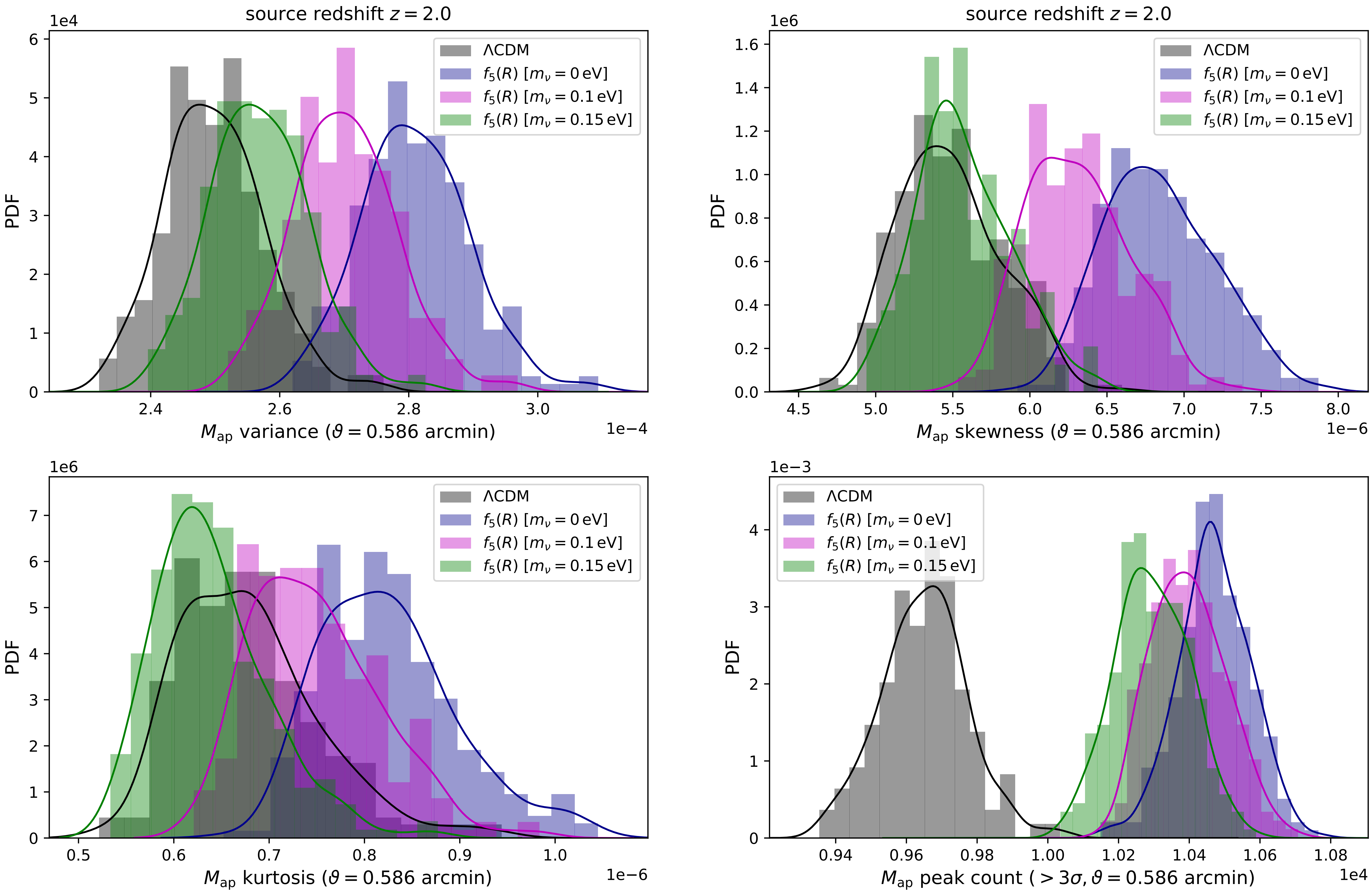
The f(R)-massive neutrinos degeneracy

Peel et al. 2018



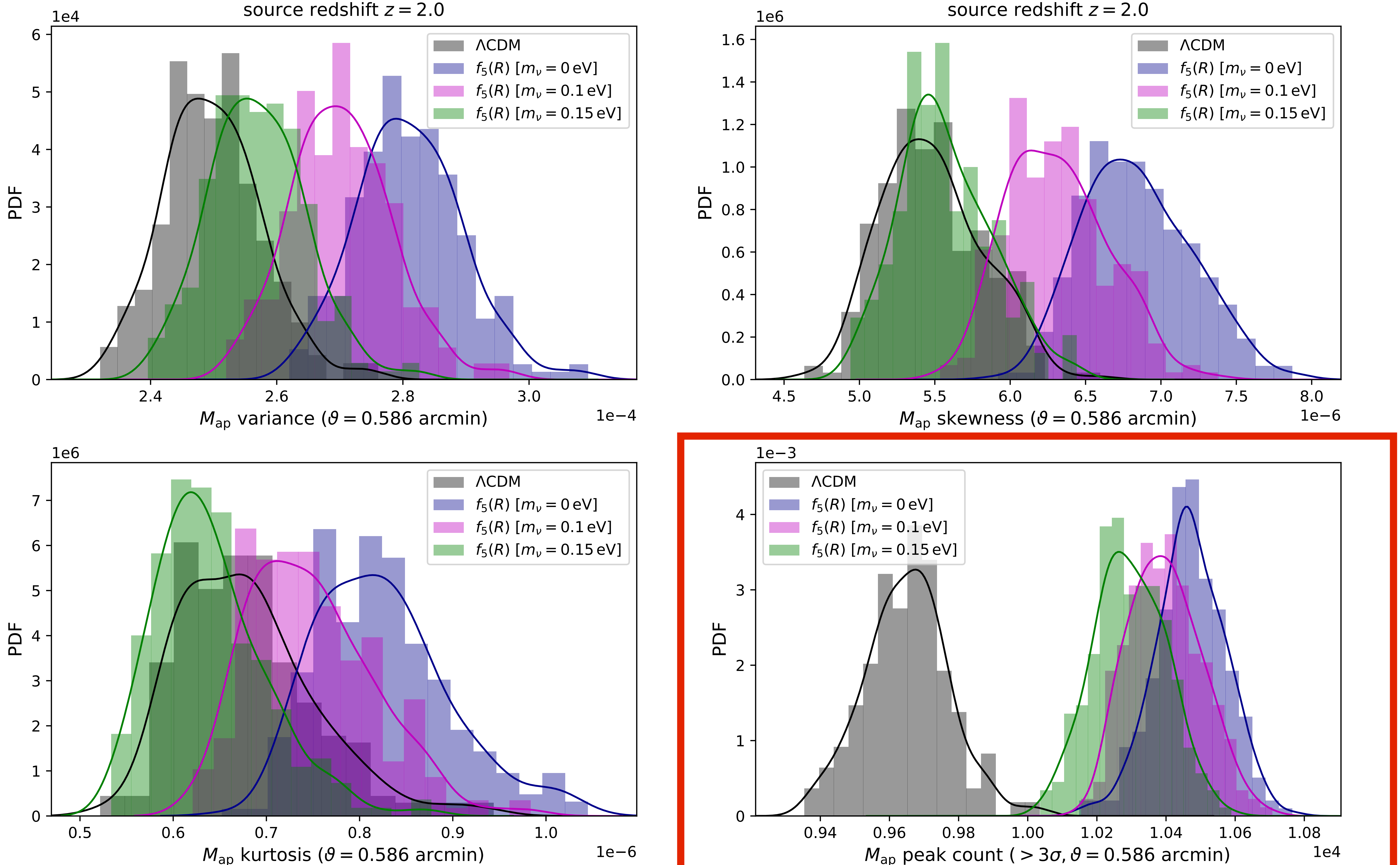
The f(R)-massive neutrinos degeneracy

Peel et al. 2018



The f(R)-massive neutrinos degeneracy

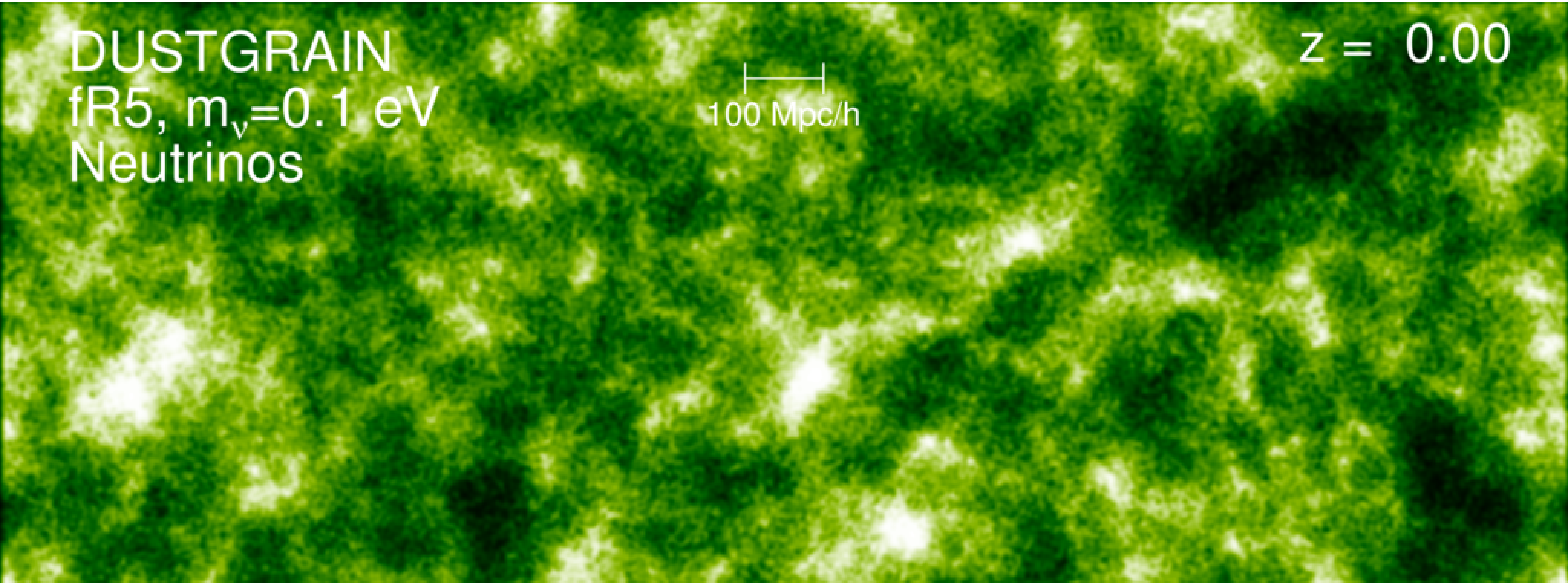
Peel et al. 2018



Summary and conclusions

- Cosmological simulations are a **necessary tool** to predict the properties and the evolution of **cosmological observables**
- Newtonian simulations can very well describe the evolution of the universe over a wide range of scales (large-scale relativistic effects can be included in post-processing for scales close to the horizon)
- **Different methods** have been developed to solve for the self-gravitational evolution of a system of N particles representing (i.e. sampling) the cosmic density field: **PM, Tree, TreePM, Multigrid**
- These methods may need to be **combined and /or modified to include additional physics** beyond the standard LCDM model for which they have been developed, such as non-trivial Dark Energy and Modified Gravity models
- Extended simulations **may be the only way** to predict characteristic features of such models allowing **to distinguish them from standard LCDM**


DUSTGRAIN
fR5, m_ν=0.1 eV
CDM
z = 0.00
100 Mpc/h
Thank you!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