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CMB Bispectrum Estimation Using Modal

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Introduction

CMB measurements have entered an era of percentage level accuracy and have given tightest constraints on cosmological parameters thus far. While the ΛCDM model remains consistent with data, the vanilla big bang model falls short in explaining issues like the flatness, relic or monopole, horizon, and scale invariance problems. Cosmic inflation has emerged as a widely accepted theory to address these challenges.

In inflationary theories, a crucial prediction is the departure from Gaussian statistics. In Gaussian scenarios, Wick's theorem simplifies the statistics of a_{lm}'s to the power spectrum C_l. However, non-Gaussianities introduce higher-order statistics in the CMB. To assess the statistical properties of CMB anisotropies in such cases, one must consider the three-point correlation function, known as the *bispectrum* of primordial curvature perturbations:

$$\langle\zeta(\boldsymbol{k}_1)\zeta(\boldsymbol{k}_2)\zeta(\boldsymbol{k}_3)\rangle = (2\pi)^3\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)B(k_1,k_2,k_3).$$

In assessing different inflation models, we will look at their predicted *shape functions*:

$$S(k_1,k_2,k_3) = (k_1k_2k_3)^2\,B(k_1,k_2,k_3).$$

Primordial non-Gaussianities will leave imprints on the CMB bispectrum:

$$\boxed{\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle = \mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}b_{l_1l_2l_3},}$$

where

$$b_{l_1l_2l_3} := \left(\frac{2}{\pi}\right)^3\iiiint \mathrm{d}r r^2 S(k_1,k_2,k_3)\prod_{i=1}^{3}j_{l_i}(k_ir)\Delta_{l_i}(k_i)\mathrm{d}k_i$$

is the *reduced bispectrum* and $\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}$ is the Gaunt integral.

Optimal Estimator

The observable Universe is only one realization, so instead of ensemble averaging, one must construct a suitable bispectrum estimator. This task becomes increasingly more difficult for higher order statistics.

Direct computation of the unbiased estimator

$$\hat{B}_{l_1l_2l_3} = \sum_{m_1m_2m_3}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}$$

requires $\mathcal{O}(l^5_{max})$ complexity. For temperature maps such as Planck with $l_{max}=2000$ the expensiveness of this procedure renders the computation of the estimator impossible even with the most modern computers. Furthermore, the signal to noise ratio in the CMB bispectrum is too low to permit model-independent detection of individual multipole components.

Instead, we measure the magnitude of the i^{th} theoretical bispectrum shape present in the signal using parameters $f^{(i)}_{\text{NL}}$:

$$B(k_1,k_2,k_3) = \sum_i f^{(i)}_{\text{NL}}B^{(i)}(k_1,k_2,k_3) \qquad \textit{(primordial)}$$

$$a_{l_1m_1}a_{l_2m_2}a_{l_3m_3} = \sum_i f^{(i)}_{\text{NL}}\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}b^{(i)}_{l_1l_2l_3} + \mathcal{E}^{m_1m_2m_3}_{l_1l_2l_3} \qquad \textit{(late time)}$$

Assuming weak non-Gaussianity and uncorrelated shapes, the single model f_{NL} estimator reduces to the least squares fit:

$$\boxed{\hat{f}_{NL} = \frac{1}{N}\sum_{l_jm_j}\frac{\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}b_{l_1l_2l_3}}{C_{l_1}C_{l_2}C_{l_3}}[a_{l_1m_1}a_{l_2m_2}a_{l_3m_3} - 3\langle a_{l_1m_1}a_{l_2m_2}\rangle^G a_{l_3m_3}],}$$

where N is the normalization:

$$N = \sum_{l_jm_j}\frac{h^2_{l_1l_2l_3}b^2_{l_1l_2l_3}}{C_{l_1}C_{l_2}C_{l_3}}$$

and the theoretical error on $\hat{f}_{\text{NL}}$ is

$$\sigma(\hat{f}_{\text{NL}}) = \sqrt{\frac{6}{N}}.$$

Tetrapyd Geometry

CMB bispectrum is defined in a region of l-space, called a *tetrapyd*, satisfying:

$$\mathcal{V}_t = \begin{cases} l_1+l_2\geq l_3 & \text{(triangle condition)}\\ l_1,l_2,l_3\leq l_{max} & \text{(resolution)}\\ l_1+l_2+l_3 = 2n, n\in\mathbf{N} & \text{(parity)}\end{cases}$$

and similarly first two conditions apply to the primordial bispectrum space $\mathcal{V}_k$. All our functions will be defined within $\mathcal{V}_{k/l}$. In particular, we define the inner product

$$\langle\bar{f},\bar{g}\rangle_l = \sum_{l_1,l_2,l_3\in\mathcal{V}_T}\bar{w}(l_1,l_2,l_3)\bar{f}(l_1,l_2,l_3)\bar{g}(l_1,l_2,l_3)$$

$$\langle f,g\rangle_k = \int_{\mathcal{V}_k}w(k_1,k_2,k_3)f(k_1,k_2,k_3)g(k_1,k_2,k_3)\mathrm{d}\mathcal{V}_k$$

with weights

$$\bar{w}(l_1,l_2,l_3) = \left(\frac{h_{l_1l_2l_3}}{v_{l_1}v_{l_2}v_{l_3}}\right)^2 \quad\text{and}\quad w(k_1,k_2,k_3) = \frac{1}{k_1+k_2+k_3},$$

where we use a separable approximation

$$\sqrt{h^2_{l_1l_2l_3}} \approx v_{l_1}v_{l_2}v_{l_3} \equiv [(2l_1+1)(2l_2+1)(2l_3+1)]^{1/6}.$$

Modal Decomposition

Computation of b_{l_1,l_2,l_3} can be sped up if $B(k_1,k_2,k_3) = X(k_1)Y(k_2)Z(k_3)$ (cf. KSW estimator). The modal approach[1] expands a general shape in terms of the *separable basis functions*:

$$S(k_1,k_2,k_3) = \sum_{n=0}\alpha_nQ_n(k_1,k_2,k_3), \qquad S_{l_1l_2l_3} = \sum_{n=0}\bar{\alpha}_n\bar{Q}_n(l_1,l_2,l_3),$$

where $S_{l_1,l_2,l_3} = \sqrt{\frac{v^2_{l_1}v^2_{l_2}v^2_{l_3}}{C_{l_1}C_{l_2}C_{l_3}}}b_{l_1l_2l_3}$ is a separable CMB shape function. We express basis functions as a symmetrised sum of products of polynomials:

$$Q_n(k_1,k_2,k_3) = \frac{1}{6}[q_p(k_1)q_r(k_2)q_s(k_3) + \text{perms.}] \equiv q_{\{p}q_rq_{s\}},$$

$$\bar{Q}_n(l_1,l_2,l_3) = \frac{1}{6}[\bar{q}_p(l_1)\bar{q}_r(l_2)\bar{q}_s(l_3) + \text{perms.}] \equiv \bar{q}_{\{p}\bar{q}_r\bar{q}_{s\}}$$

with a one-to-one mapping between indices $n\leftrightarrow\{prs\}$. The functions are *not* orthonormal:

$$\langle Q_p,Q_r\rangle_k \equiv \gamma_{pr}\neq\delta_{pr} \quad\text{and}\quad \langle\bar{Q}_p,\bar{Q}_r\rangle_l \equiv \bar{\gamma}_{pr}\neq\delta_{pr}.$$

The coefficients can then be extracted as

$$\alpha_n = \sum_p\gamma^{-1}_{np}\langle Q_p,S\rangle_k \quad\text{and}\quad \bar{\alpha}_n = \sum_p\bar{\gamma}^{-1}_{np}\langle\bar{Q}_p,\bar{S}\rangle_l.$$

Projecting primordial Q-basis to late time,

$$\tilde{Q}_n(l_1,l_2,l_3) = \sqrt{\frac{v^2_{l_1}v^2_{l_2}v^2_{l_3}}{C_{l_1}C_{l_2}C_{l_3}}}\int_{\mathcal{V}_k}Q_n(k_1,k_2,k_3)\Delta_{l_1l_2l_3}(k_1,k_2,k_3)\mathrm{d}\mathcal{V}_k,$$

which is related to the CMB basis via Γ matrix,

$$\Gamma_{np} = \sum_r\bar{\gamma}^{-1}_{nr}\langle\bar{Q}_r,\tilde{Q}_p\rangle_l,$$

provides projection of primordial α_n's to the corresponding CMB coefficients $\bar{\alpha}_n$:

$$\bar{\alpha}_n = \sum_p\Gamma_{np}\alpha_p.$$

Next, we separate the triple sum over l,m into a product of *maps*:

$$\bar{M}_p(\Omega) = \sum_{l,m}\bar{q}_p(l)\frac{a_{lm}}{v_l\sqrt{C_l}}Y_{lm}(\Omega)$$

to obtain

$$\bar{\beta}_n = \int\mathrm{d}\Omega\bar{M}_{\{p}\bar{M}_r\bar{M}_{s\}} - \int\mathrm{d}\Omega\left\langle\bar{M}^G_{\{p}\bar{M}^G_r\right\rangle\bar{M}_{s\}} \equiv \bar{\beta}^{\text{cub}}_n - 3\bar{\beta}^{\text{lin}}_n.$$

The basis can be orthonormalized, $S = \sum_n\alpha^{\mathcal{R}}_n\mathcal{R}_n$, via Cholesky decomposition of $\gamma^{-1} = \lambda\lambda^T$, so that

$$\bar{\alpha}^{\mathcal{R}}_n = \sum_p\bar{\lambda}^{-1}_{pn}\bar{\alpha}_p \quad\text{and}\quad \bar{\beta}^{\mathcal{R}}_n = \sum_p\bar{\lambda}_{np}\bar{\beta}_p.$$

This reduces the f_{NL} estimator to

$$\boxed{\hat{f}_{\text{NL}} = \frac{\sum_n\bar{\alpha}_n\bar{\beta}_n}{\sum_{np}\bar{\alpha}_n\bar{\gamma}_{np}\bar{\alpha}_p} = \frac{\sum_n\bar{\alpha}^{\mathcal{R}}_n\bar{\beta}^{\mathcal{R}}_n}{\sum_n\bar{\alpha}^{\mathcal{R}}_n\bar{\alpha}^{\mathcal{R}}_n}.}$$

With polarization included[2]:

$$\hat{f}_{\text{NL}} = \frac{\sum_n\left(\bar{\alpha}^{TTT}_n\beta^{TTT}_n + 3\bar{\alpha}^{TTE}_n\beta^{TTE}_n + 3\bar{\alpha}^{TEE}_n\beta^{TEE}_n + \bar{\alpha}^{EEE}_n\beta^{EEE}_n\right)^{\mathcal{R}}}{\sum_n\left(\bar{\alpha}^{TTT^2}_n + 3\bar{\alpha}^{TTE^2}_n + 3\bar{\alpha}^{TEE^2}_n + \bar{\alpha}^{EEE^2}_n\right)^{\mathcal{R}}}.$$

Reconstructing Primordial Shapes

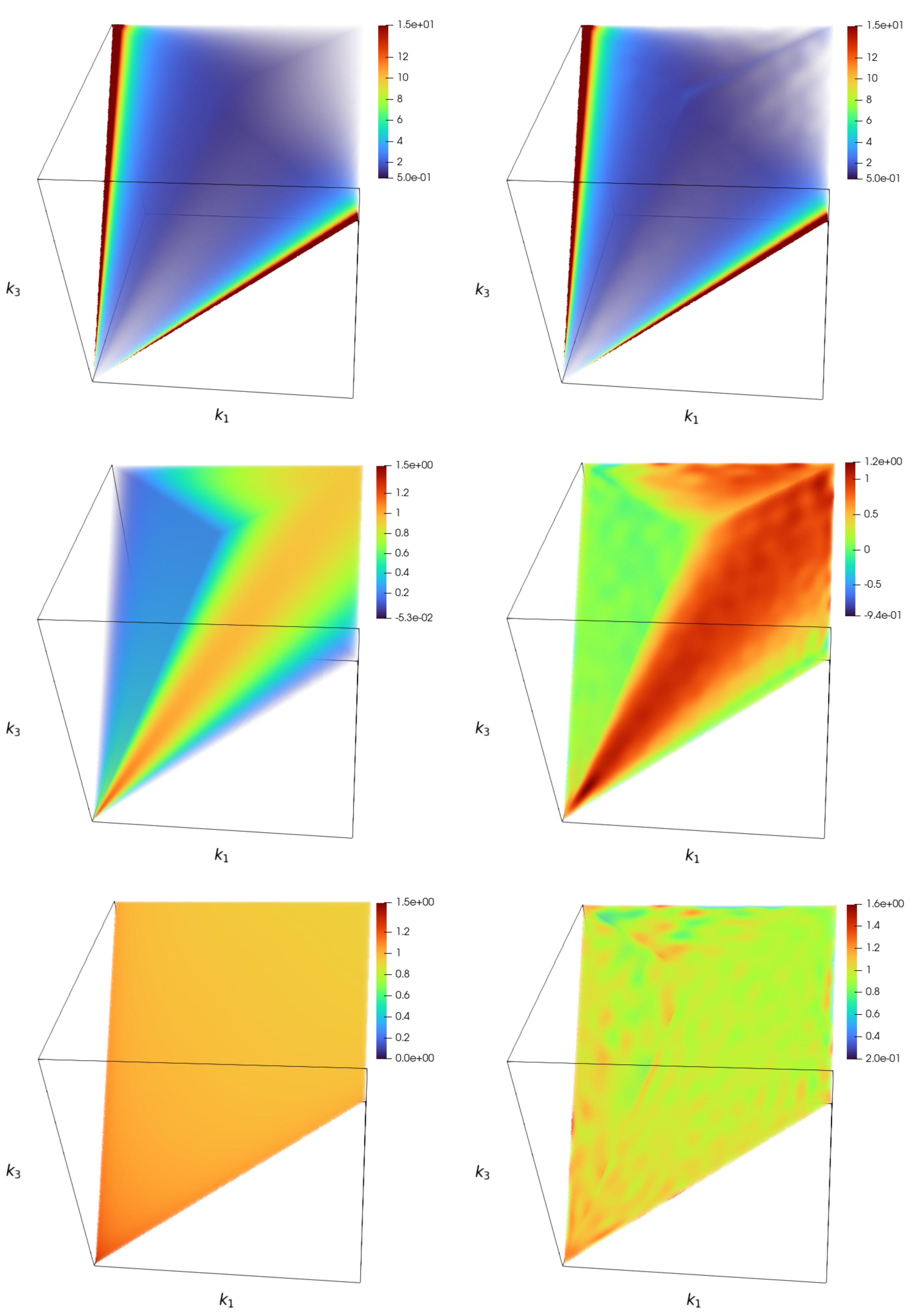

Figure 1. Theoretical (left) and reconstructed (right) *local*, *equilateral* and *constant* shape with recovered behaviour in the squeezed and equilateral limit.

CMB Bispectrum Results

For full Planck analysis up to 2000 terms were used but here we used 500 terms for computational effectiveness and proof of concept. Although orthogonal shape gives the most promising result, recent calculations show significant biases from cosmic infrared background (CIB) lensing[3] which are not yet accounted for.

Model	Official Planck	Results with $n=500$
DBI	46 ± 58	31 ± 57
Equilateral	34 ± 67	15 ± 66
Local	-0.6 ± 6.4	-1.1 ± 6.7
Orthogonal	-26 ± 43	-33 ± 46
Model	Official Planck	Results with $n=500$
DBI	14 ± 38	20 ± 42
Equilateral	-4 ± 43	-0.4 ± 47.9
Local	-2.0 ± 5.0	-0.4 ± 5.6
Orthogonal	-40 ± 24	-45 ± 27

Table 1. Constraints on shapes for `SMICA` T-only maps and `SMICA` $T+E$ maps.

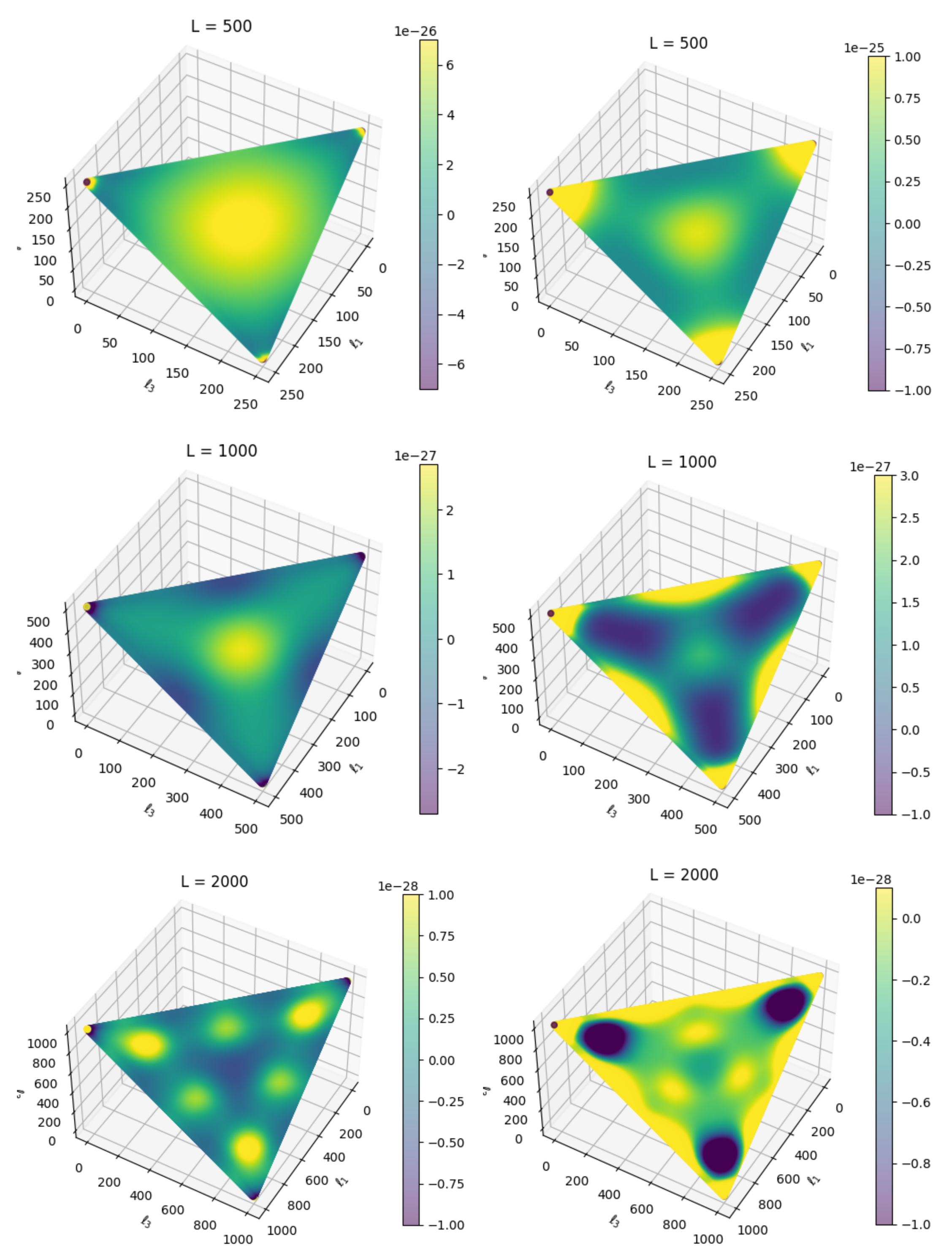

Figure 2. DBI (left) and orthogonal (right) CMB bispectrum shapes for different $L = l_1+l_2+l_3$ slices.

Conclusion and Future Goals

The pipeline is set and running for both T and $T+E$ modes and has been implemented on a new supercomputer architecture compared to the previous use. We successfully reconstruct primordial and late time bispectrum with constraints consistent with Planck.

Primordial non-Gaussianities might remain undetected if an incorrect theoretical template was used. It is, therefore, crucial to investigate as many different (uncorrelated) templates as one can possibly motivate from the inflationary models and look for new theories. Current efforts are put into faithfully reconstructing the oscillating shape $S = \sin\left((k_1+k_2+k_3)\,\omega+\phi\right)$, for which current reconstruction breaks down at frequencies $\omega\gtrsim 300$[4].

Next goal is to extend the pipeline to Simons Observatory with $l_{max}=5000$. Increase in resolution requires more modes in shape reconstruction. Expanding the number of terms in the basis risks it becoming degenerate due to numerical integration errors. It will be necessary to consider novel approaches in constructing the basis and computing inner product of their elements. Particularly promising is the analytical, rather than numerical, computation of $\langle q_i,q_j\rangle_{k/l}$ up to an arbitrary n_{max} developed by Baker, D.

References

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[3] William Coulton, Alexander Miranthis, and Anthony Challinor. "Biases to primordial non-Gaussianity measurements from CMB secondary anisotropies". In: *Monthly Notices of the Royal Astronomical Society* 523.1 (May 2023), pp. 825–848. DOI: `10.1093/mnras/stad1305`. URL: `https://doi.org/10.1093%2Fmnras%2Fstad1305`.

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