Decoding the Path Integral: Resurgent Asymptotics and Extreme Quantum Field Theory

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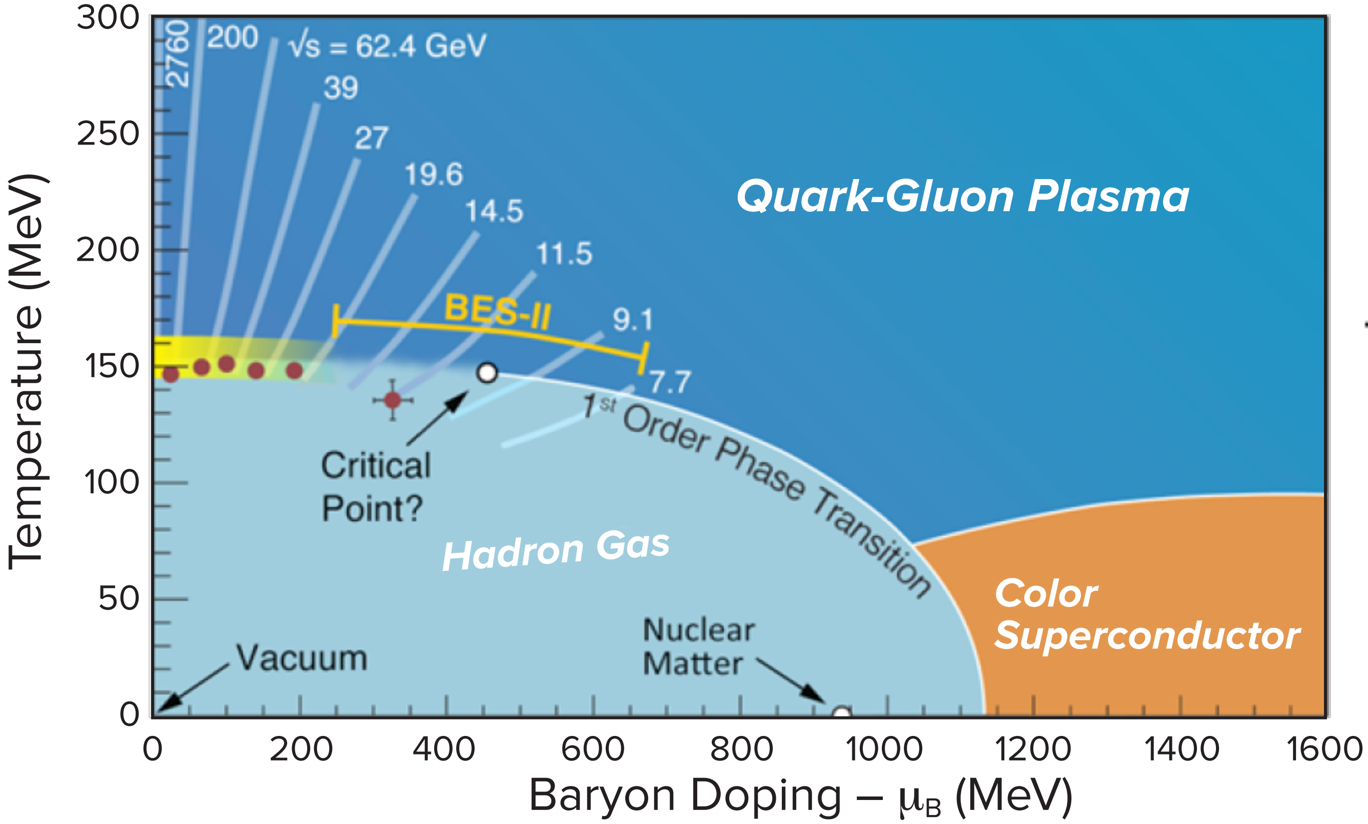
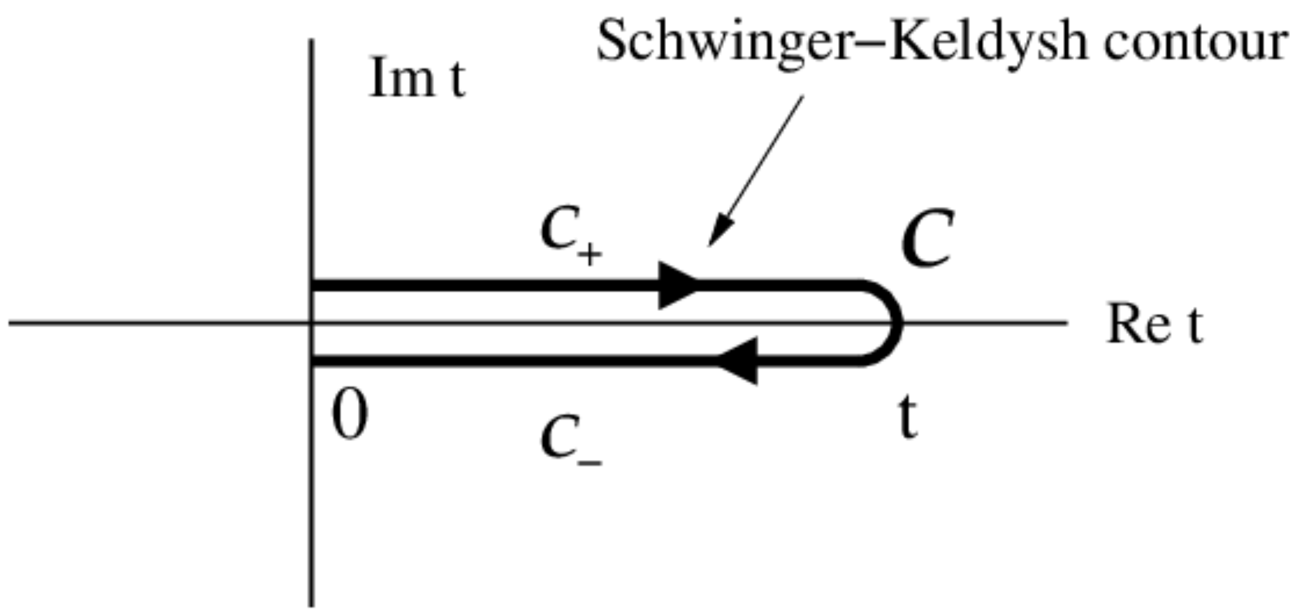
University of Connecticut

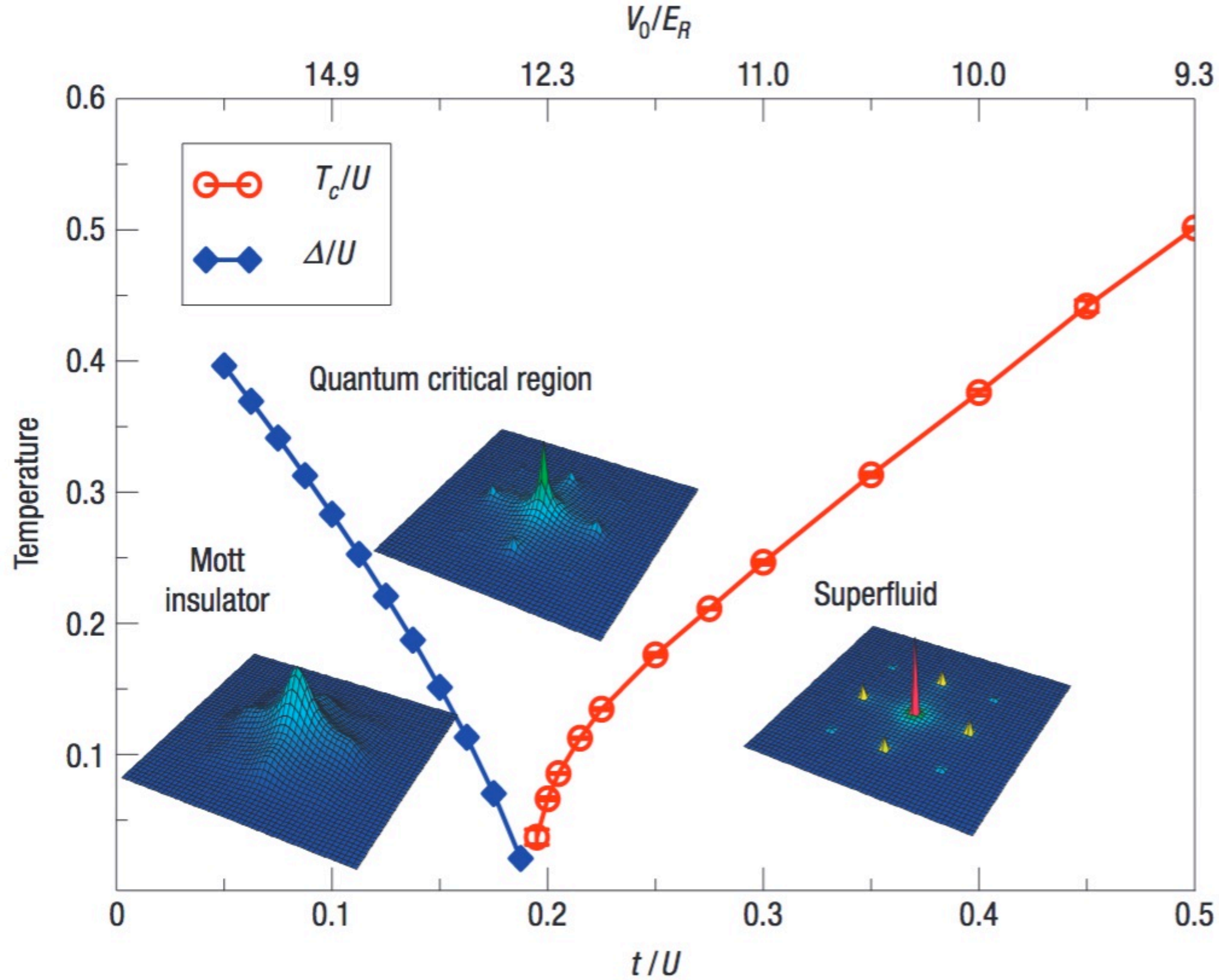
review: arXiv:1601.03414 ; winter school lectures

Quantum Systems in Extreme Conditions (QSEC2019)
Heidelberg, September 2019

Physical Motivation







Physical Motivation: Quantum Physics in Extreme Conditions

- QCD phase diagram
- non-equilibrium physics at strong-coupling
- (quantum) phase transitions in cold atom systems
- quantum systems in extreme background fields
- transition to hydrodynamics
- quantum gravity

extreme systems are extremely difficult to analyze quantitatively

extreme = strongly-coupled; high density; ultra-fast driving; ultra-cold; strong fields; strong curvature; heavy ion collisions; …

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- non-perturbative semi-classical methods: “instantons"
- non-perturbative numerical methods: Monte Carlo
- asymptotics

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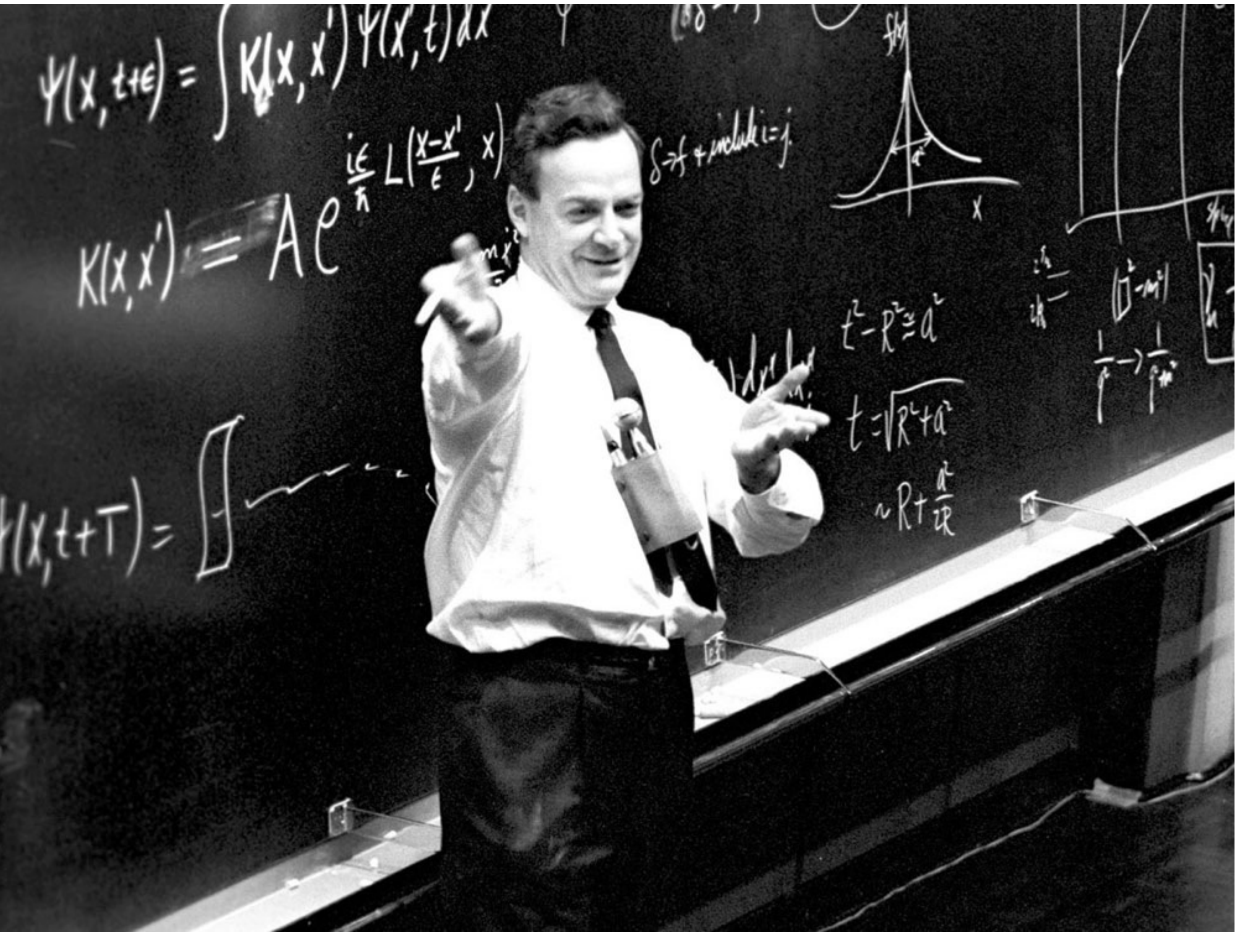
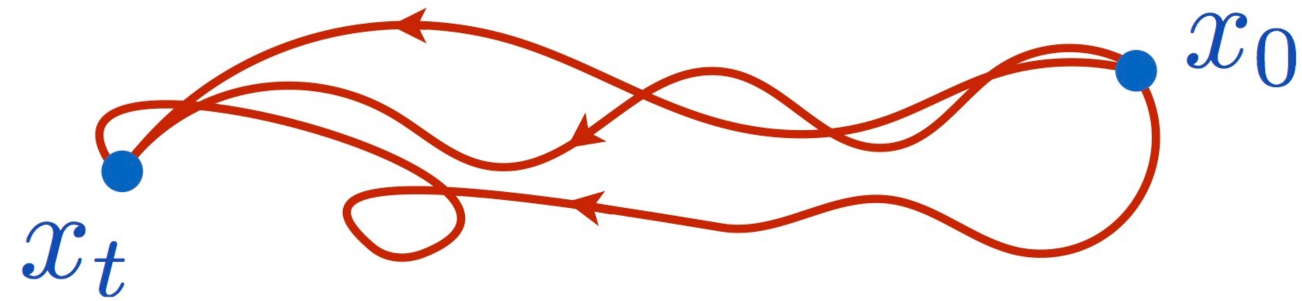
technical problem: what does a quantum path integral really mean?

The Feynman Path Integral



$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



QM: $$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

QFT: $$\int \mathcal{D}A(x^\mu)\, \exp\left[\frac{i}{g^2} S\left[A(x^\mu)\right]\right]$$

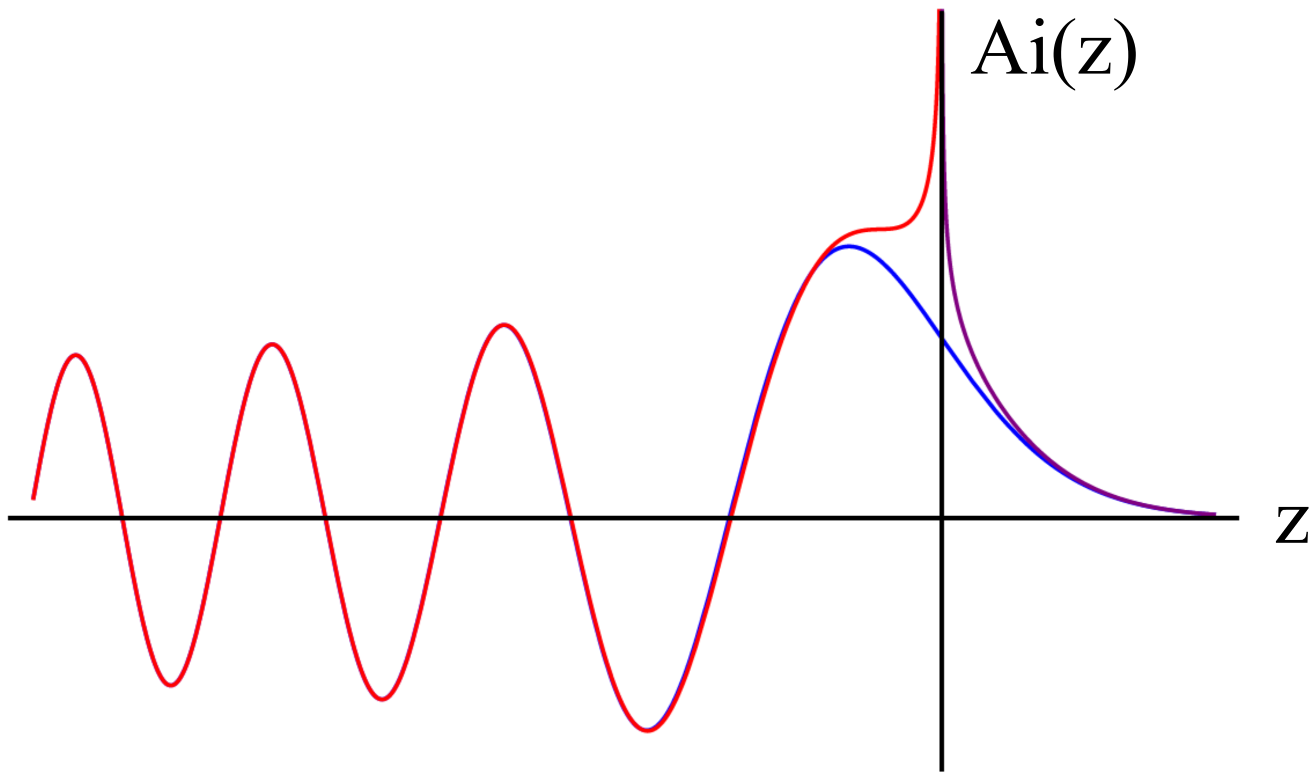
- stationary phase approximation: classical physics
- quantum perturbation theory: fluctuations about trivial saddle point
- other saddle points: non-perturbative physics
- resurgence: saddle points are related by analytic continuation, so perturbative and non-perturbative physics are *unified*

Resurgence in Classical Optics

Stokes and supernumerary rainbows …

photo: R. Bishop

Stokes and the Airy Function: "Stokes Phenomenon"



$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}x^3 + z\,x\right)}dx \sim \begin{cases} \dfrac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}\,z^{1/4}} & , \quad z \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-z)^{1/4}} & , \quad z \to -\infty \end{cases}$$

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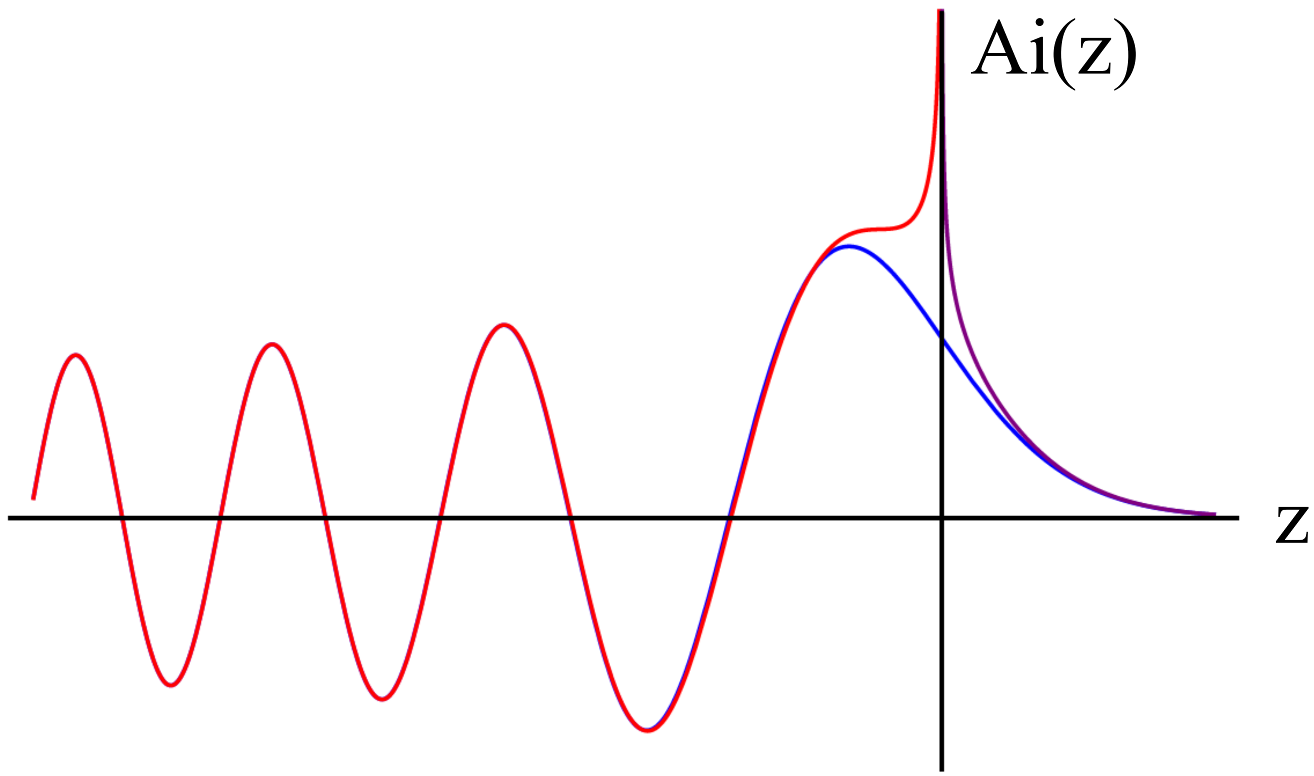

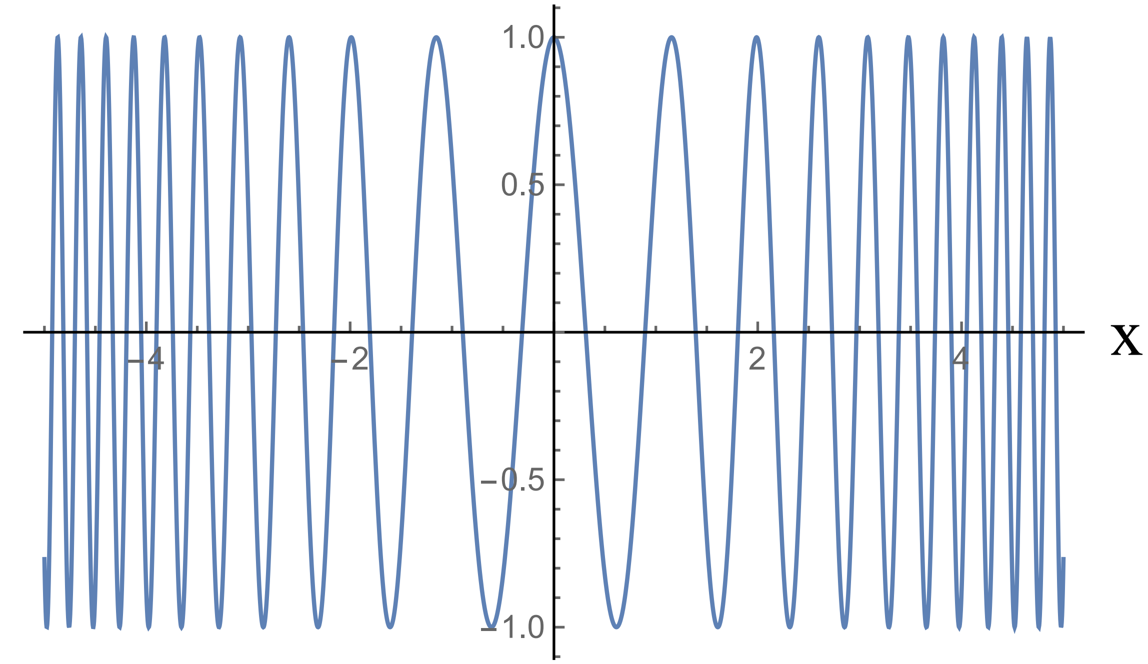

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- integral <u>cannot</u> be evaluated without contour deformation
- "Stokes transition" at z=0
- fluctuation expansions about saddles <u>must</u> be divergent, and <u>must</u> be related
- underlies optics and WKB analysis

Analytic Continuation of Path Integrals

since we need complex analysis and contour deformation to make sense of oscillatory integrals, it is natural to explore similar methods for (infinite dimensional) path integrals

$$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar}\, S[x(t)]\right] \longleftrightarrow \int \mathcal{D}x(t)\, \exp\left[-\frac{1}{\hbar}\, S[x(t)]\right]$$

why is this important ?

- Minkowski versus Euclidean path integrals
- "sign problem" of finite density quantum systems
- non-equilibrium quantum transport
- real-time quantum dynamics
- Euclidean/Minkowski quantum gravity
- phase transitions as the Stokes phenomenon (Lee-Yang)

Resurgent Asymptotics

resurgence: "new" idea in mathematics

Dingle 1960s, Ecalle, 1980; Stokes 1850

perturbative series $\longrightarrow$ "trans-series"

$$f(\hbar) = \sum_p c_{[p]}\, \hbar^p \quad \longrightarrow \quad f(\hbar) = \sum_k \sum_p \sum_l c_{[kpl]}\, e^{-\frac{k}{\hbar}}\, \hbar^p\, (\ln \hbar)^l$$

physics:
- unifies perturbative and non-perturbative physics
- QFT "multi-instanton calculus"

math:
- trans-series is well-defined under analytic continuation
- expansions about different saddles are related
- exponentially improved asymptotics
- with iterations, all problems are solved by trans-series
- dynamical systems, differential equations, fluids, …

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Ecalle, 1980

theorem/claim: this structure occurs for all "natural" problems

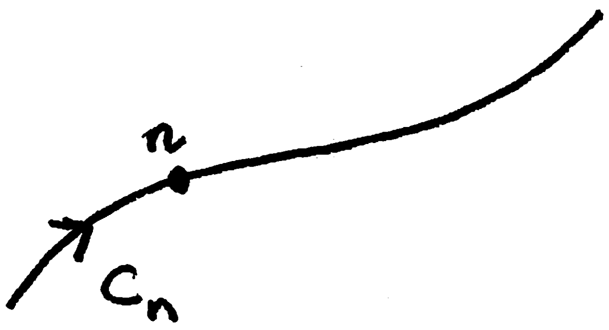
Resurgence in Exponential Integrals

steepest descent integral through saddle point "n":

$$I^{(n)}(\hbar) = \int_{C_n} dx\, e^{\frac{i}{\hbar}\, f(x)} = \frac{1}{\sqrt{1/\hbar}}\, e^{\frac{i}{\hbar}\, f_n}\, T^{(n)}(\hbar)$$

all fluctuations beyond the Gaussian approximation:

$$T^{(n)}(\hbar) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, \hbar^r$$



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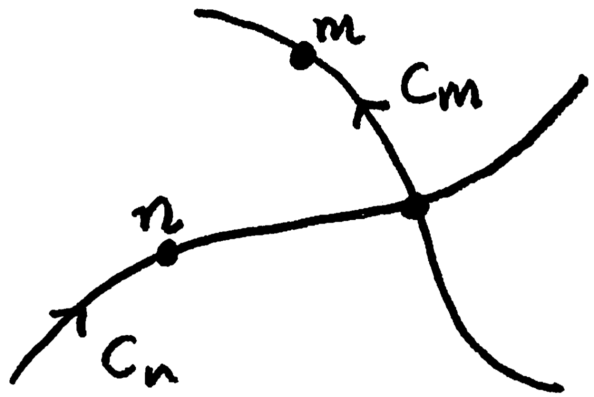

straightforward complex analysis implies:

universal large orders of fluctuation coefficients: $(F_{nm} \equiv f_m - f_n)$

$$T_r^{(n)} \sim \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(\pm 1)}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \dots \right]$$

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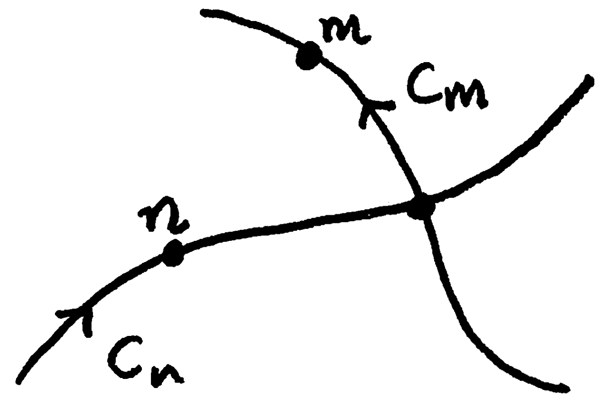

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<u>fluctuations about different saddles are quantitatively related !!!</u>

Resurgence in Exponential Integrals

canonical example: Airy function: 2 saddle points

$$T_r^{\pm} = (\pm 1)^r \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(r+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^r r!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

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large orders of fluctuation coefficients:

$$T_r^{+} \sim \frac{(r-1)!}{(2\pi)\left(\frac{4}{3}\right)^r}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(r-1)} + \left(\frac{4}{3}\right)^2\frac{385}{4608}\frac{1}{(r-1)(r-2)} - \dots\right)$$

generic "large-order/low-order" resurgence relation

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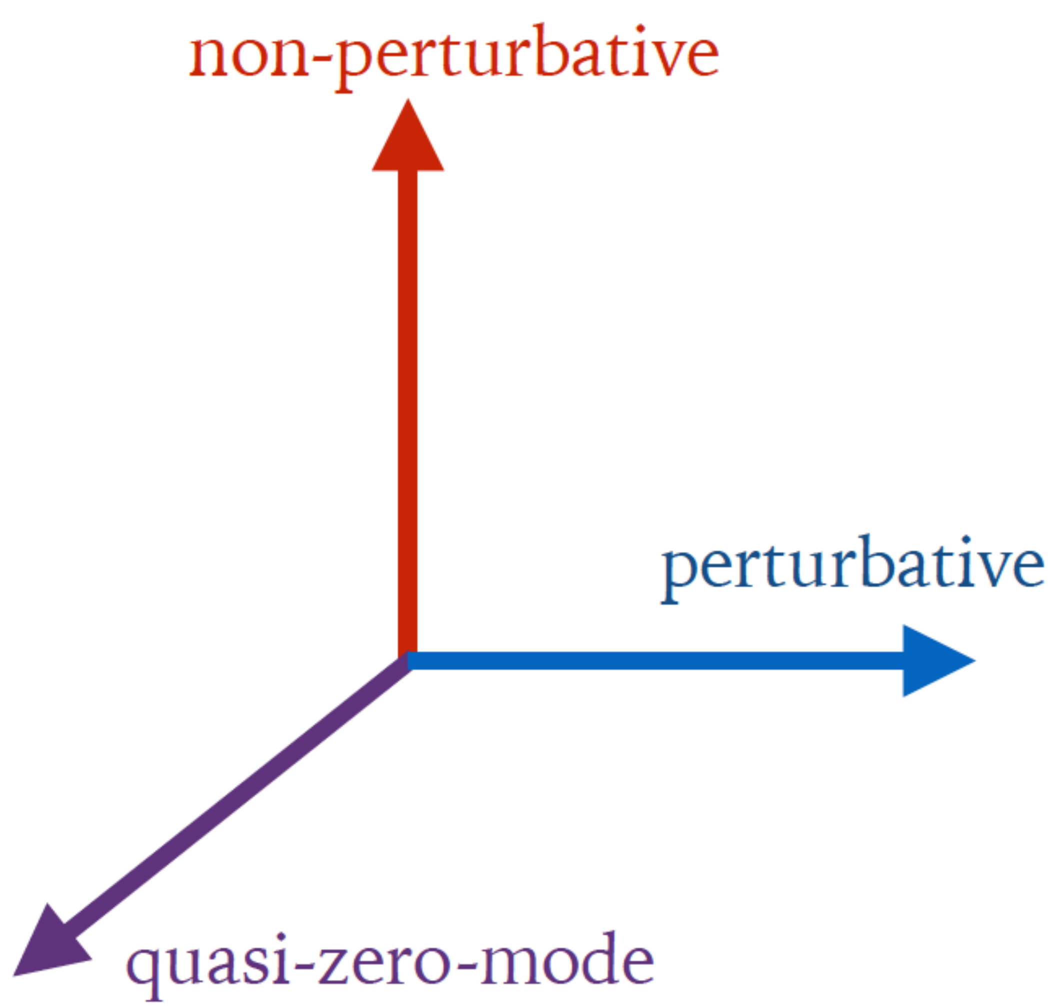
generic "large-order/low-order" resurgence relation

amazing fact: this generic large-order behavior has been observed in infinite dim. path integrals in matrix models, QM, QFT, string theory, …

the only natural way to explain this is via analytic continuation of path integrals

Decoding a Path Integral as a Trans-Series

$$\int \mathcal{D}A\, e^{\frac{i}{\hbar}S[A]} = \sum_{\text{thimbles}} e^{\frac{i}{\hbar}S[A_{\text{thimble}}]} \times (\text{fluctuations}) \times (\text{qzm})$$



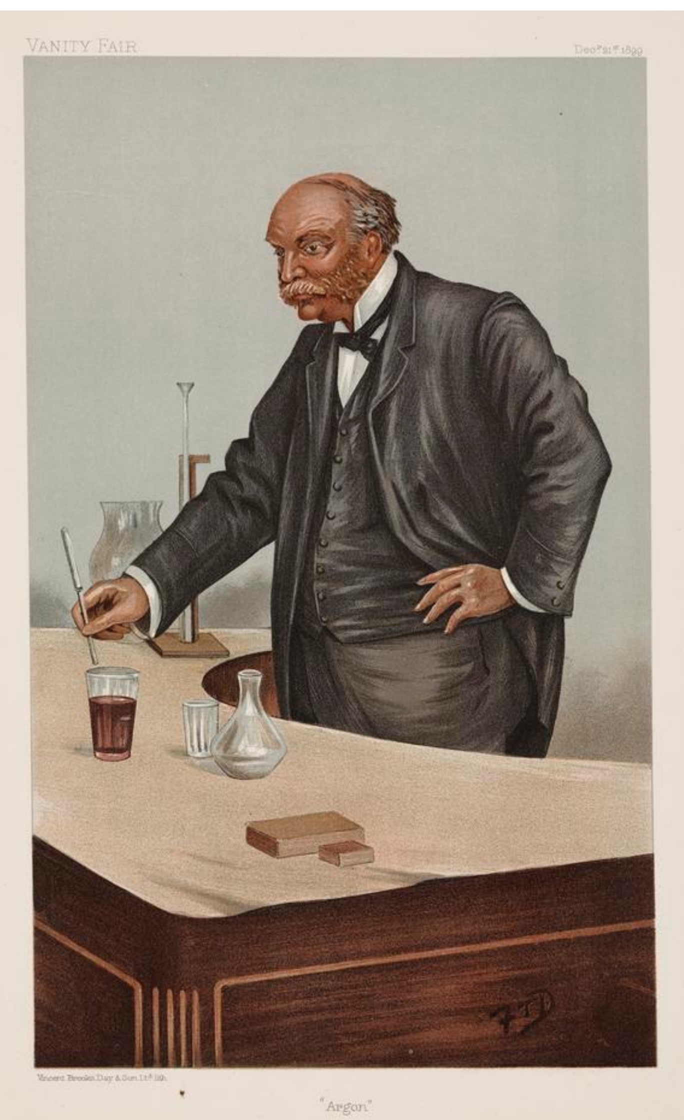
- expansions along different axes must be <u>quantitatively</u> related
- expansions about different saddles must be <u>quantitatively</u> related

Rayleigh-Schrödinger Perturbation Theory

nontrivial physics problems are rarely solvable

perturbation theory

hard problem = easy problem + “small correction”



$$(H_0 + \epsilon H')\, \psi = E\, \psi$$

$$E_1 = \langle \psi_0 \,|\, H' \,| \psi_0 \rangle$$

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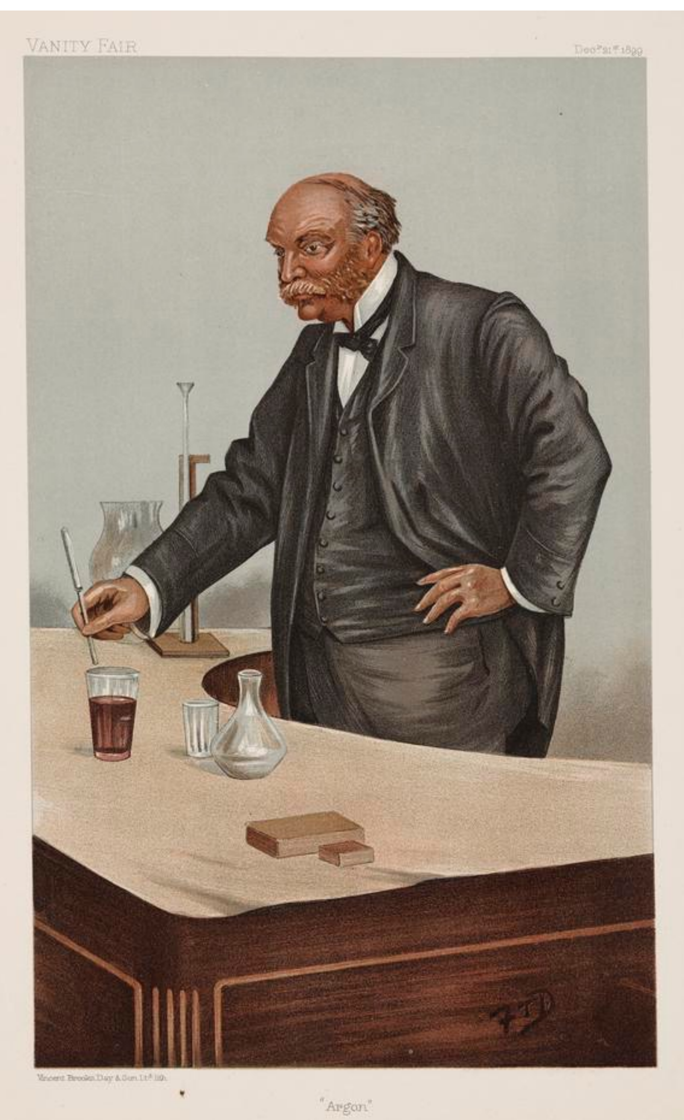


$$(H_0 + \epsilon H')\,\psi = E\,\psi$$

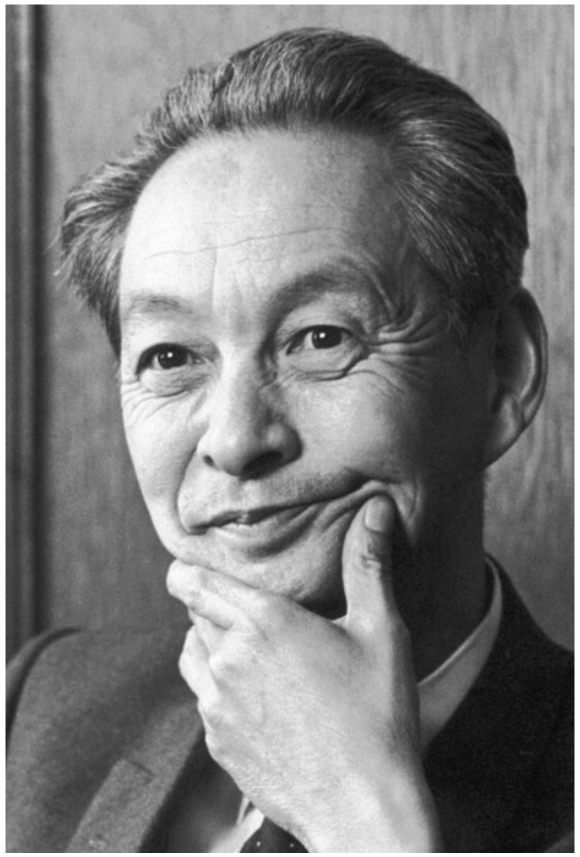
$$E_1 = \langle \psi_0 \,|\, H' \,|\psi_0\rangle$$

simple, versatile,
intuitive,
and it works

e.g. Zeeman, Stark, …

Quantum Electrodynamics (QED)

Nobel Prize 1965: Tomonaga, Schwinger, Feynman



"renormalization": finiteness of perturbation theory term-by-term

$$\left(\frac{g-2}{2}\right)_{\text{theory}} = \frac{1}{2}\frac{\alpha}{\pi} - 0.3285\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.1812\ldots\left(\frac{\alpha}{\pi}\right)^3 - 1.9106(20)\left(\frac{\alpha}{\pi}\right)^4 + 9.16(58)\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

spectacular success of perturbation theory

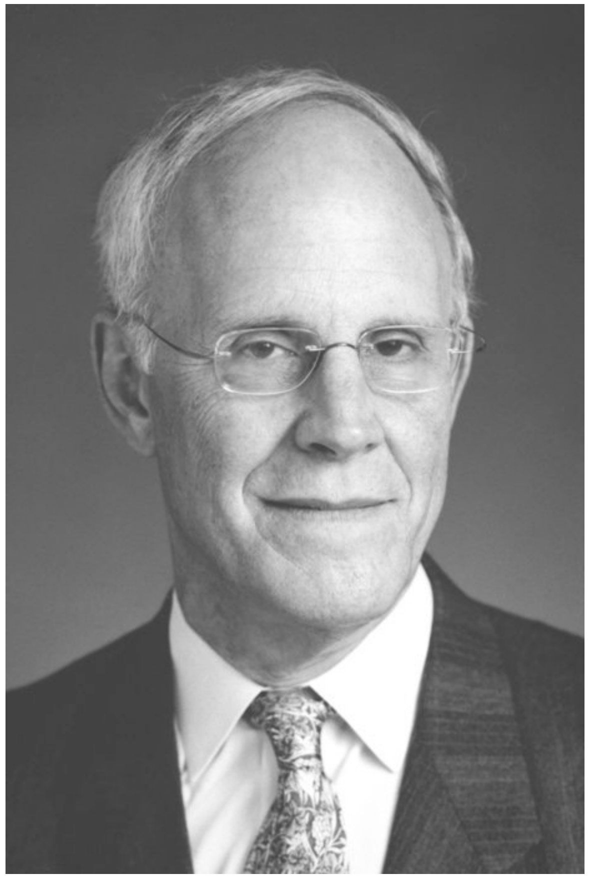
$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

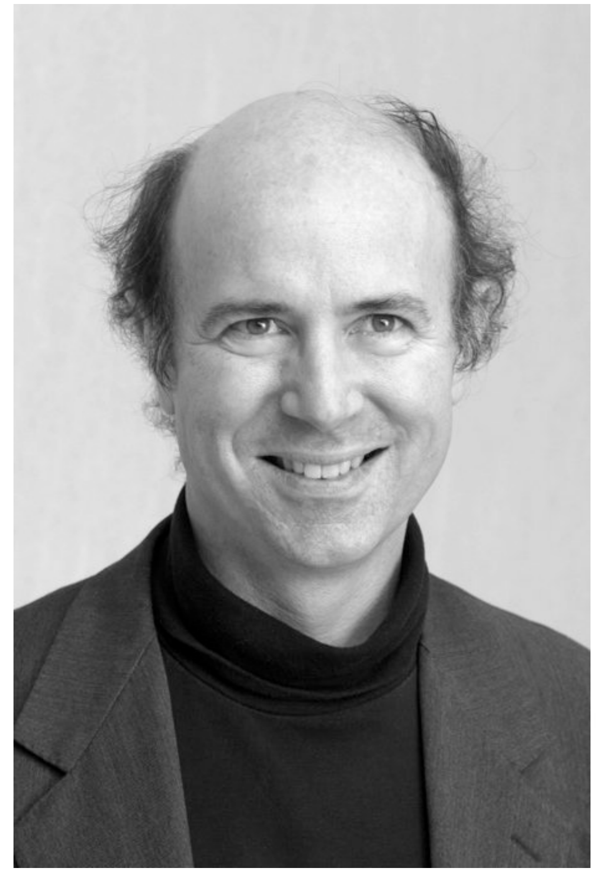
Gabrielse et al, 2008

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,181\,78(77)$$

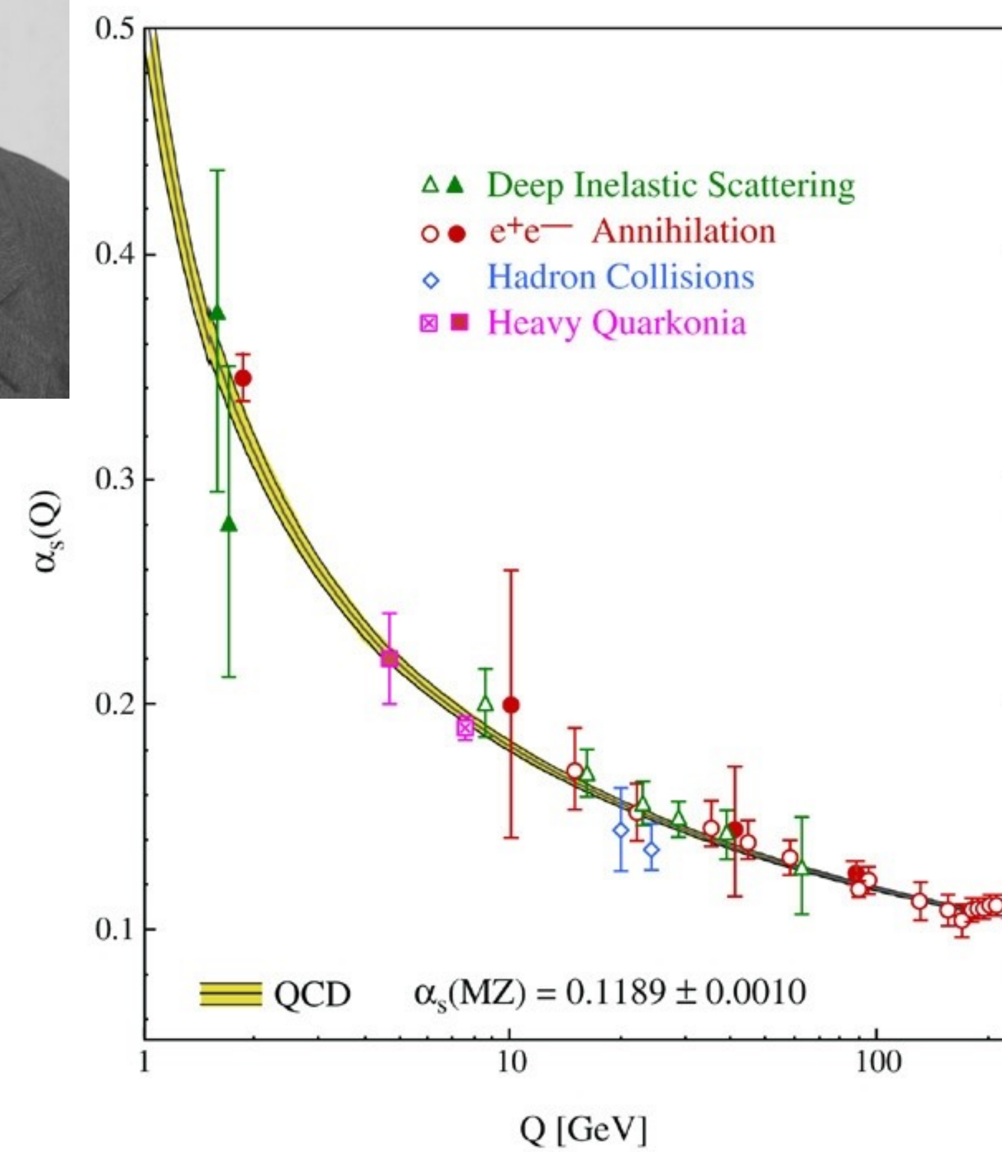
Quantum Chromodynamics (QCD)

Nobel Prize 2004: Gross, Politzer, Wilczek





"asymptotic freedom"
perturbation theory
at short distances



$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

another spectacular success of
perturbation theory

The Truth About Perturbation Theory

perturbation theory works, but it is generically divergent

this is actually a good thing !

and there is a lot of interesting physics behind this

PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

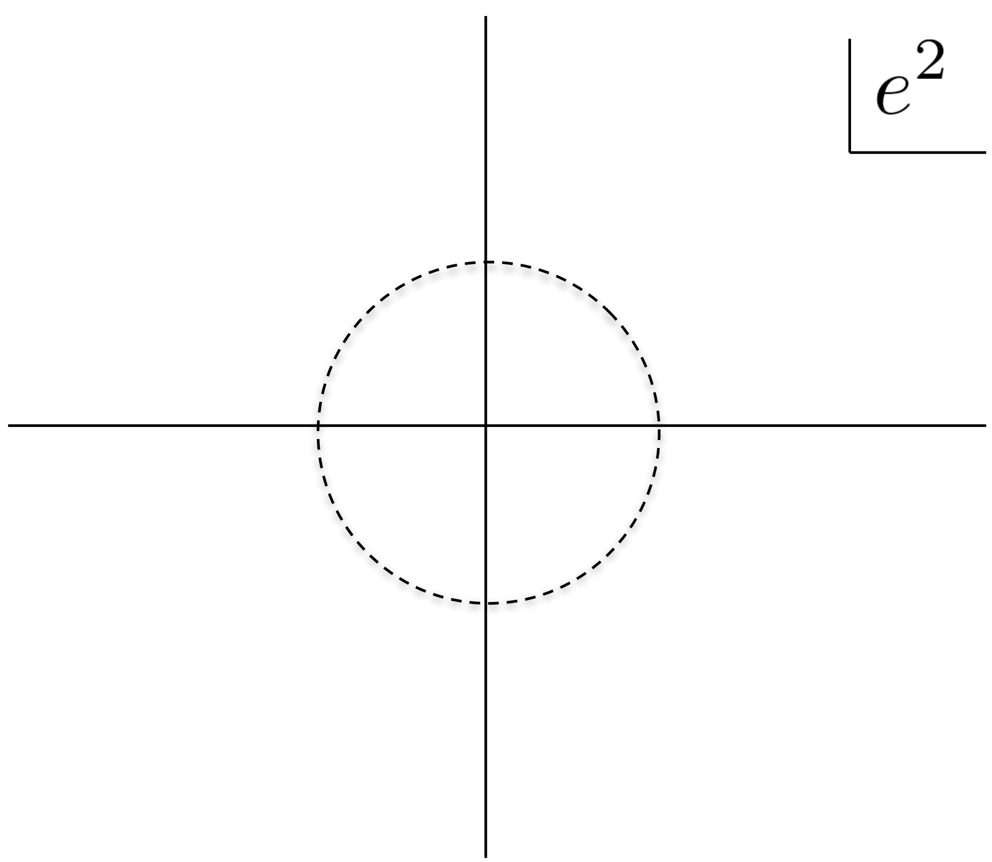
Divergence of Perturbation Theory in Quantum Electrodynamics

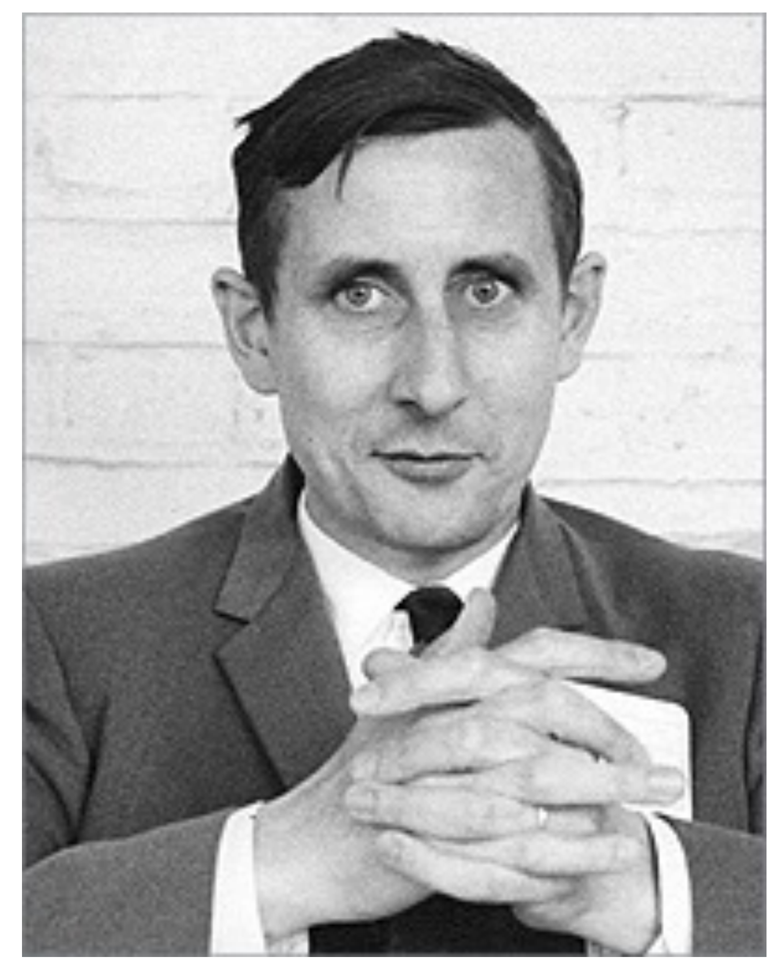
F. J. DYSON
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

$$F = a_0 + a_1 e^2 + a_2 e^4 + a_3 e^6 + \dots$$





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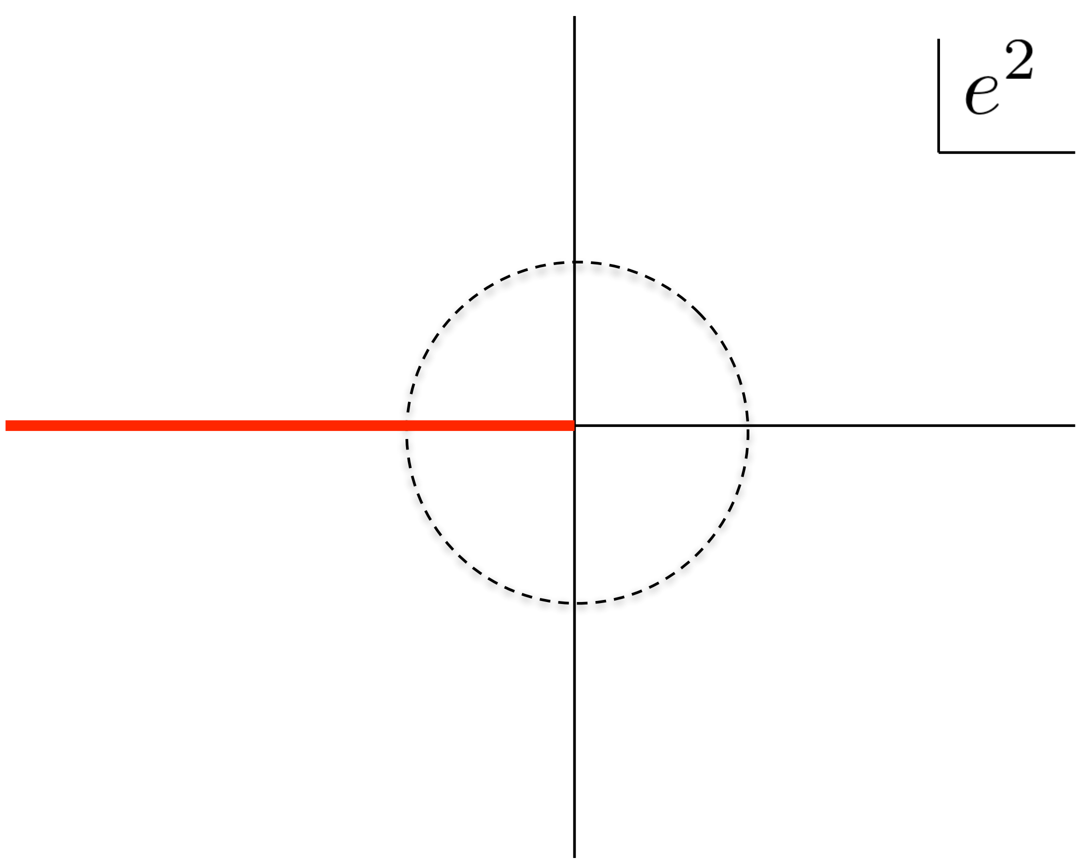
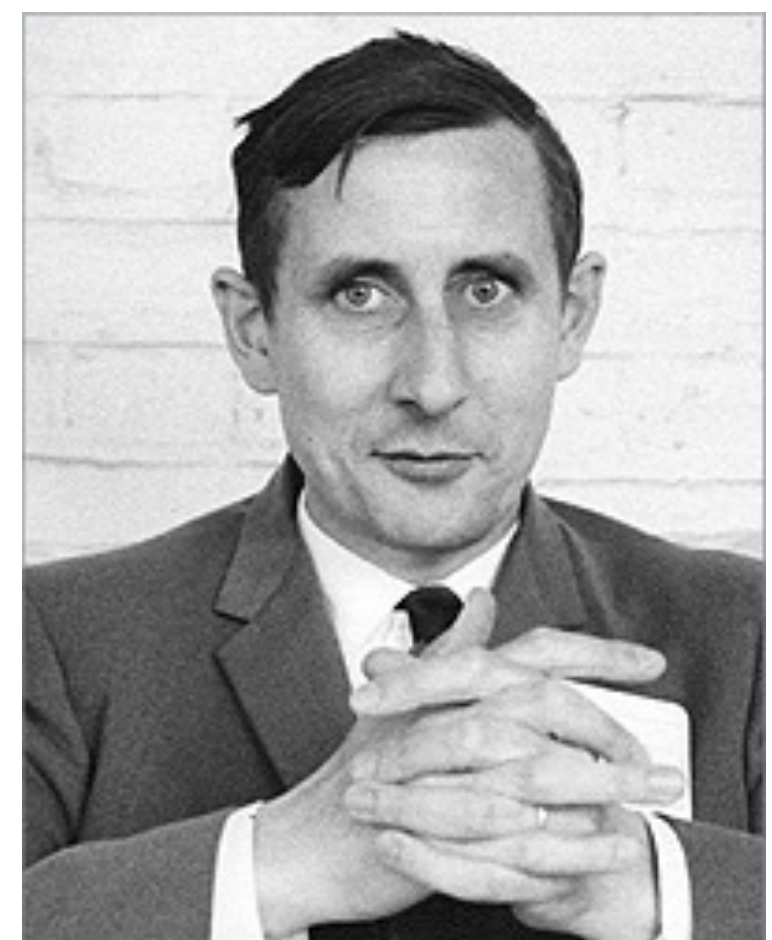
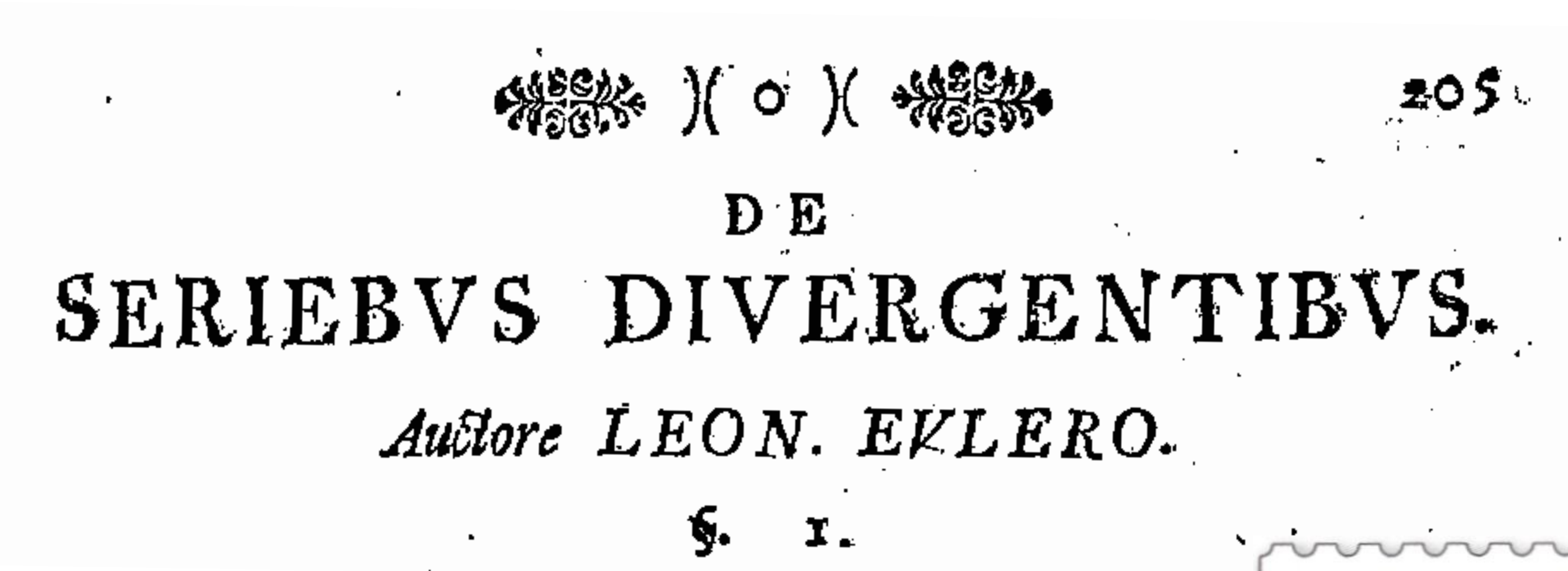


$e^2 < 0$

unstable



The Struggle to Make Sense of Divergent Series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DE

SERIEBVS DIVERGENTIBVS.

Auctore LEON. EVLERO.

§. 1.

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = ???$$

L. Euler, *De seriebus divergentibus*, Opera Omnia, I, 14, 585–617, 1760.

The Struggle to Make Sense of Divergent Series

"Borel summation"

factorial: $n! = \int_0^\infty dt\, e^{-t}\, t^n$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n n!\, x^n = \int_0^\infty dt\, e^{-t} \frac{1}{1 + x\, t}$$

convergent for all x > 0

Emile Borel

The Struggle to Make Sense of Divergent Series

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convergent for all x > 0

$$f(-x) = \sum_{n=0}^{\infty} n!\, x^n = \int_0^\infty dt\, e^{-t} \frac{1}{1-x\,t}$$

$\Rightarrow$ $\mathrm{Im}[f(-x)] \sim e^{-1/x}$

nonperturbative imaginary part !!!

Emile Borel

Zeeman & Stark Effects Revisited

Zeeman : divergent, alternating, asymptotic series

$$a_n \sim (-1)^n (2n)!$$

physics: magnetic field causes level shifts (real)

Stark : divergent, non-alternating, asymptotic series

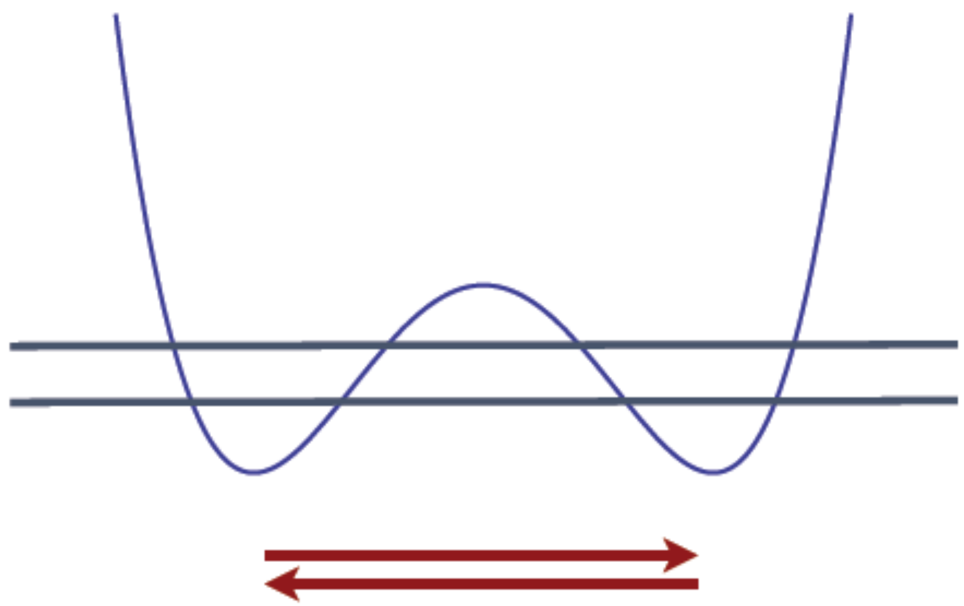
$$a_n \sim (2n)!$$

physics:
- electric field causes level shifts (real)
- and ionization (imaginary, exponentially small)

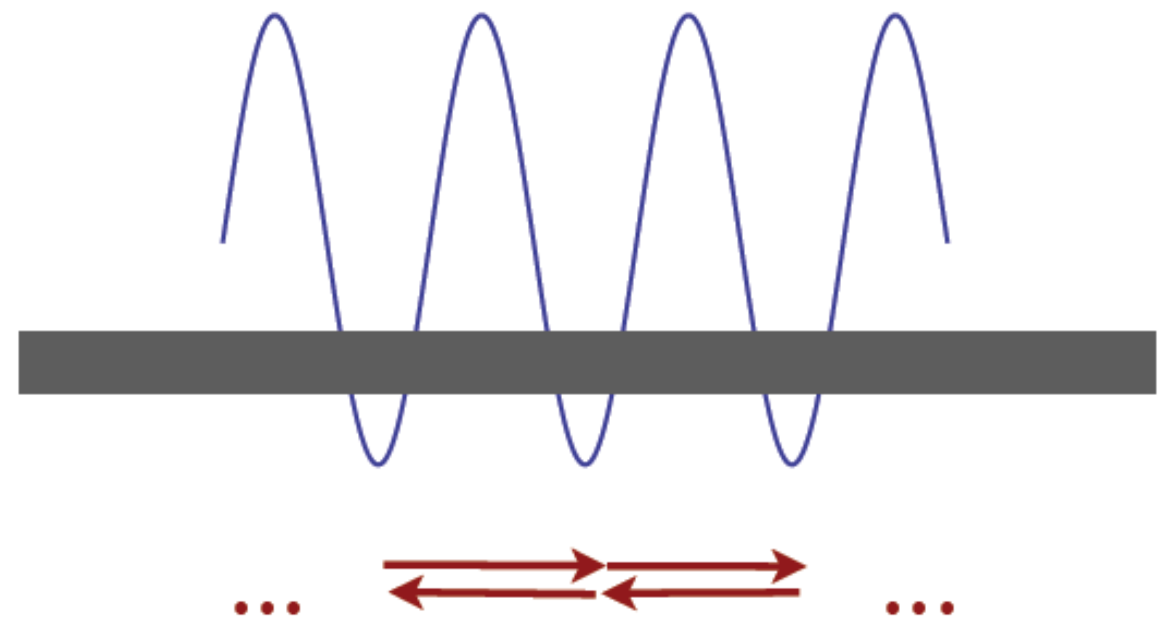
but not so fast …

the story becomes even more interesting …

Instantons and Non-Perturbative Physics



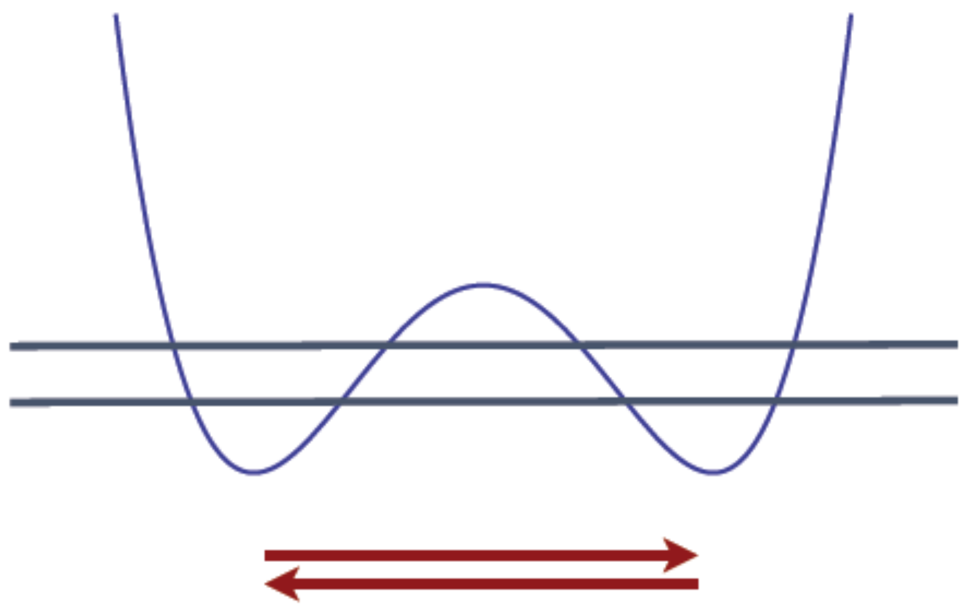
(phase transitions)



(band structure)

- exponentially small non-perturbative splitting due to tunneling
- Yang-Mills theory and QCD have aspects of <u>both</u> systems (see "renormalon" discussion later)
- physics of optical lattices and condensates

Instantons and Non-Perturbative Physics



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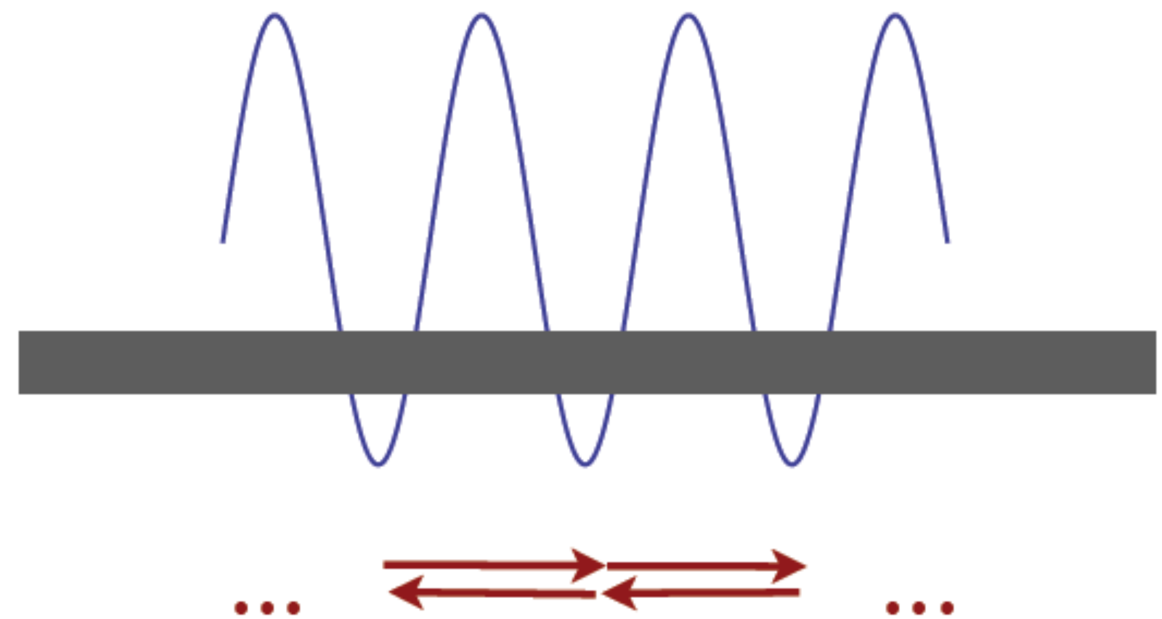

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surprise: perturbation theory is <u>non-alternating</u> divergent !

but these systems are <u>stable</u> ???

A Brilliant Resolution: "BZJ Cancelation Mechanism"

E. B. Bogomolny, 1980; J. Zinn-Justin et al, 1980

perturbation theory + Borel: $\longrightarrow$ $$+i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$$

non-perturbative instanton & anti-instanton interaction: $\longrightarrow$ $$-i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$$

unphysical imaginary parts exactly cancel !

separately, each of the perturbative and non-perturbative computations is inconsistent; but combined as a trans-series they are consistent

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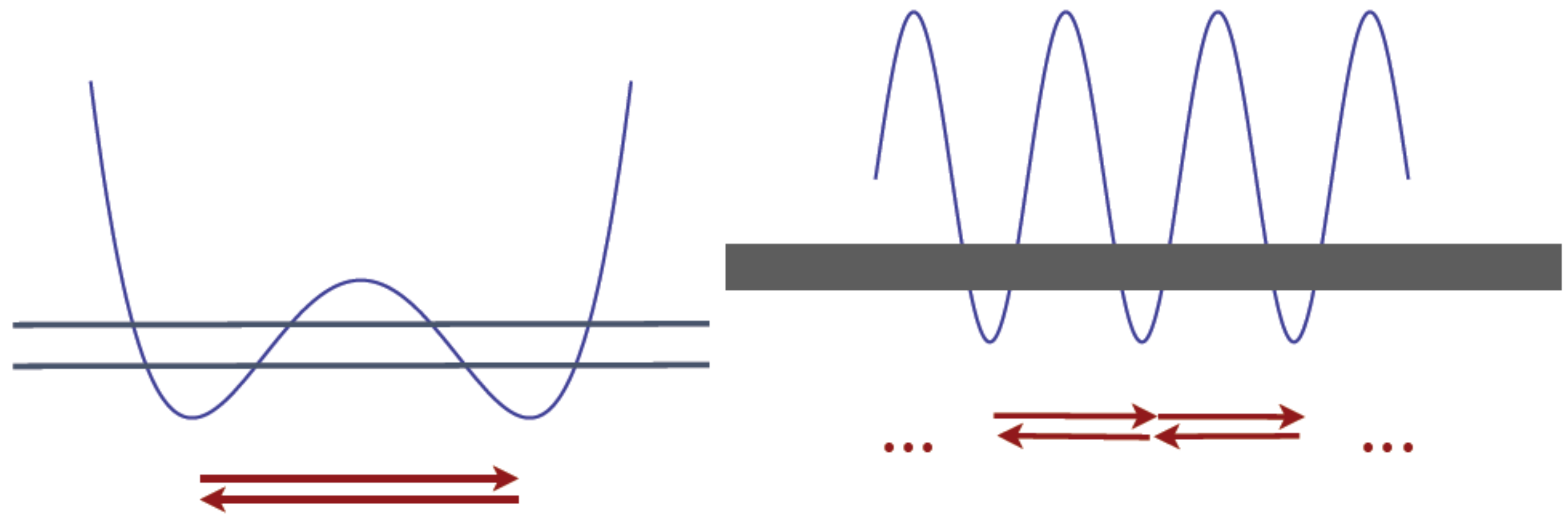
tip-of-the-iceberg: perturbative/non-perturbative relations

"Resurgence": cancelations occur to all orders; the trans-series expression for the energy is real & well-defined some extra magic…

Resurgence in Quantum Mechanical Instanton Models



trans-series for energy, including non-perturbative splitting:

$$E_\pm(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

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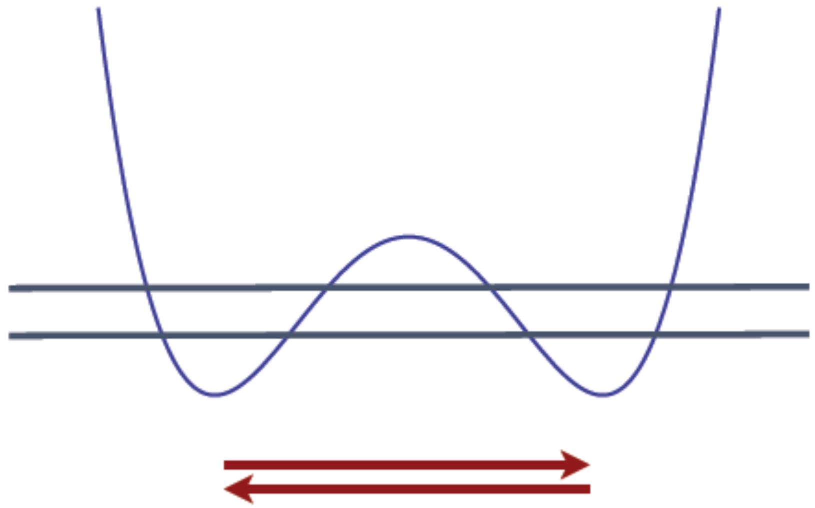

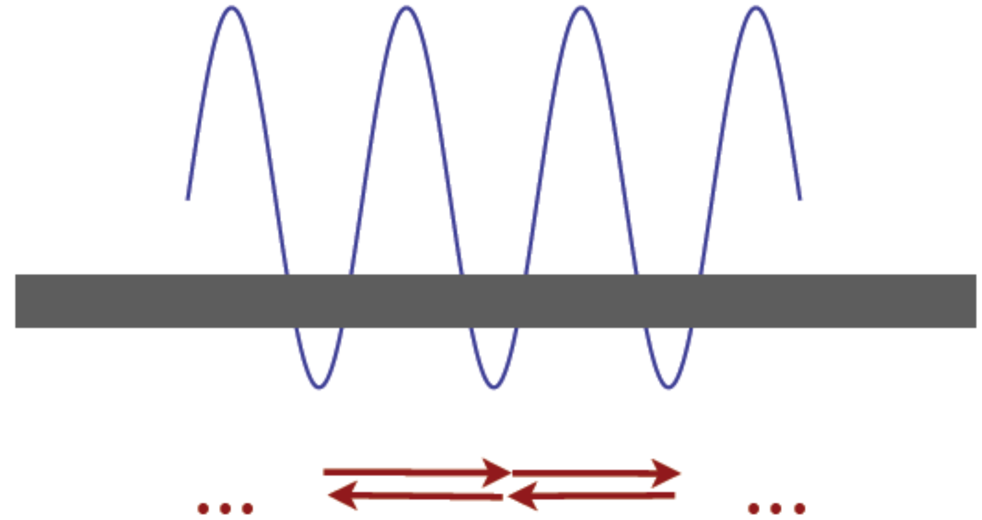

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fluctuations about first non-trivial saddle:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

perturbation theory encodes everything ! … to all orders !

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Ecalle, 1980

occurs in QM path integrals with an infinite number of saddles

Resurgence in QFT: Euler-Heisenberg Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l} \mathfrak{E}, \mathfrak{B} \quad \text{Kraft auf das Elektron.} \\ |\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.} \end{array} \right)$$

- perturbative expansion gives a doubly-divergent series
- integral is the Borel sum
- analogue of Stark ionization and Dyson’s argument
- particle production in E field <u>implies</u> series are divergent

Resurgence in QFT: Stokes Phase Transition

- Schwinger effect with monochromatic E field: $E(t) = \mathcal{E}\,\cos(\omega\, t)$
- Keldysh adiabaticity parameter: $\gamma \equiv \dfrac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB: $\Gamma_{\rm QED} \sim \exp\left[-\pi \dfrac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,g(\gamma)\right]$

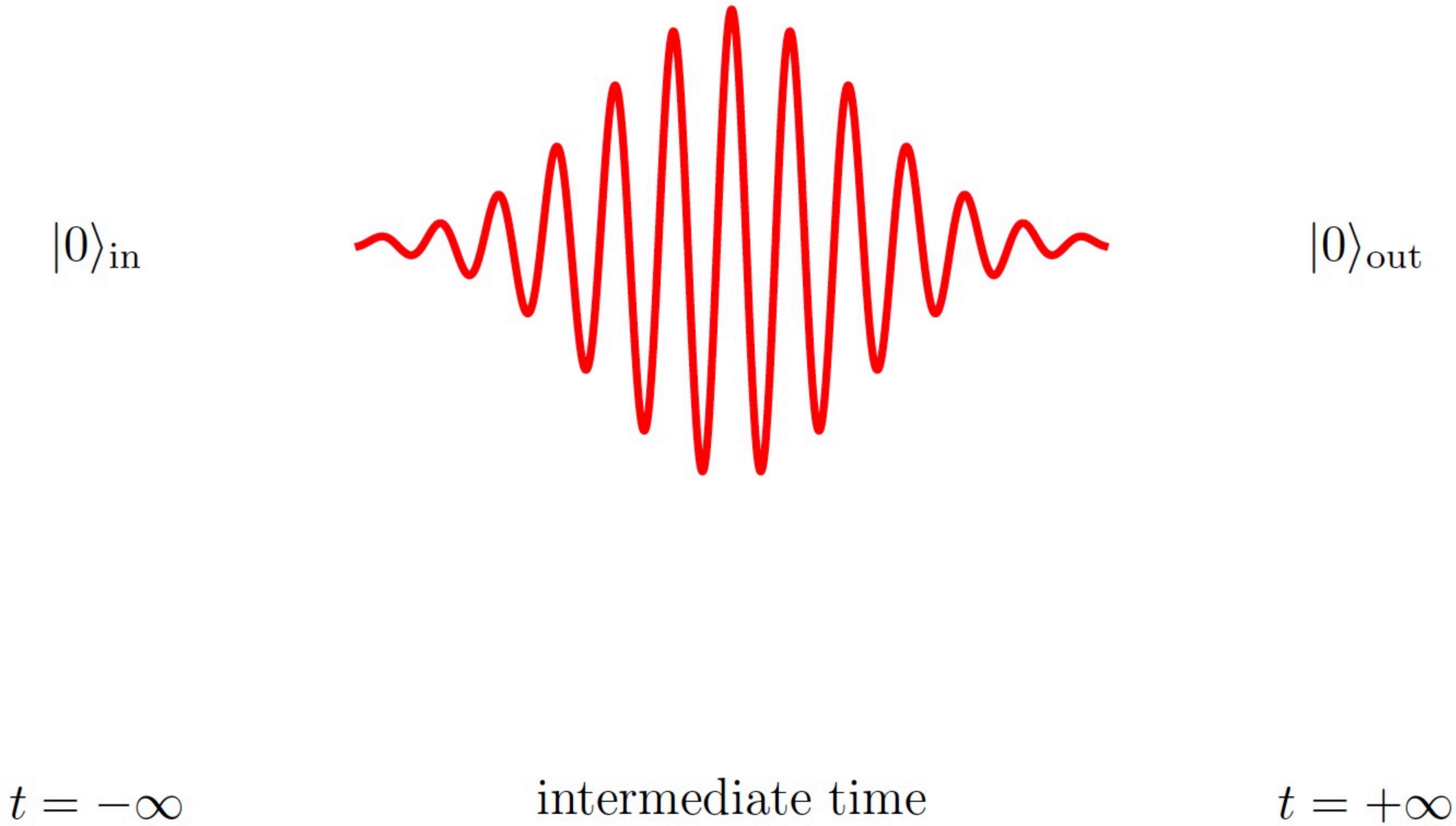
(Keldysh, 1964; Brezin/Itzykson, 1980; Popov, 1981)

$$\Gamma_{\rm QED} \sim \begin{cases} \exp\left[-\pi \frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(tunneling)} \\ \left(\frac{e\,\mathcal{E}}{m\,c\,\omega}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(multiphoton)} \end{cases}$$

- phase transition: tunneling vs. multi-photon “ionization”
- phase transition: real vs. complex instantons
- non-trivial quantum interference effects for E(t)

Resurgence in QFT: Ultra-Fast Dynamics

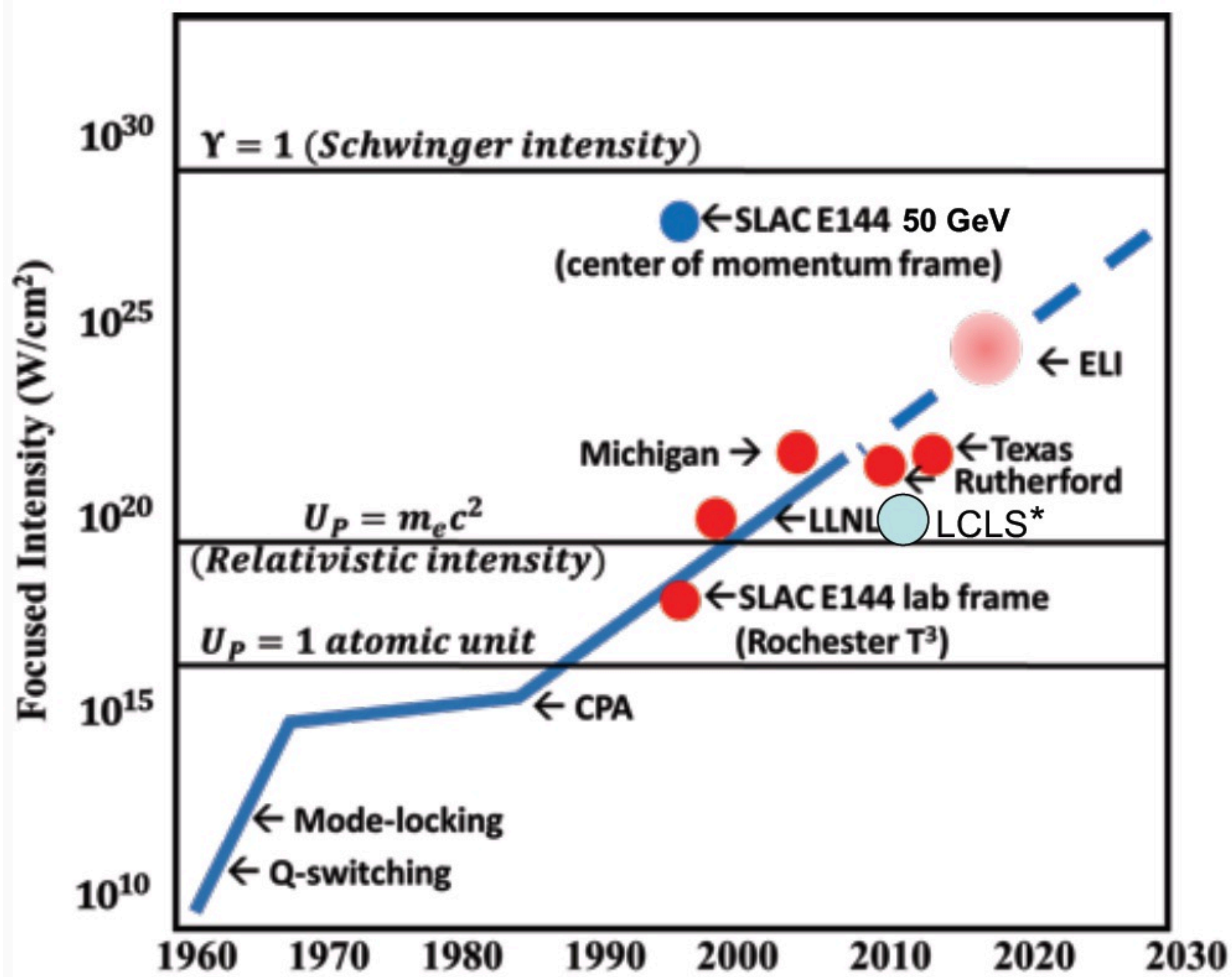
time evolution of quantum systems with ultra-fast perturbations



- adiabatic expansion is divergent
- resurgence: expansion can be (Borel) resummed to a <u>universal form</u>
- novel quantum interference effects: complex saddles
- back-reaction effects (e.g., Schwinger effect)

Probing Physics at Extreme Intensities

- Current experimental proposals: laser-laser; laser-lepton; lepton-lepton; highly-charged ions; astrophysics; …
- Important theoretical puzzles remain
- Locally constant field approximation?
- Semiclassical computations?
- Non-equilibrium physics?
- Ultra-fast dynamics



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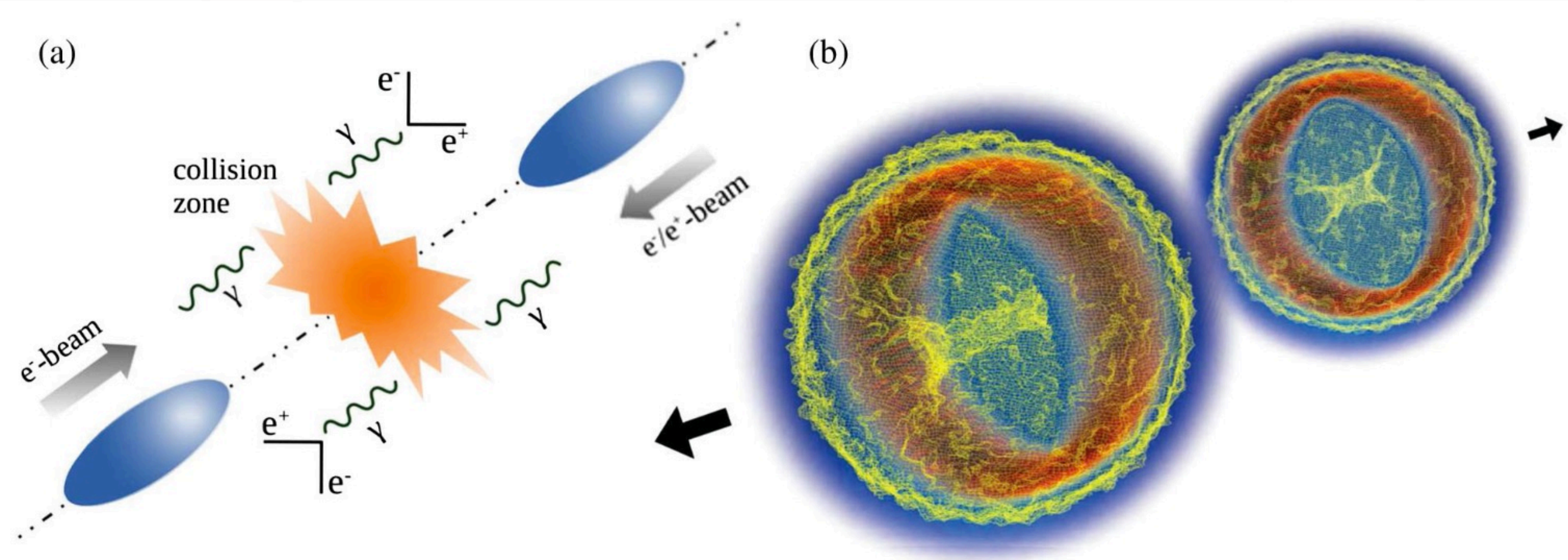
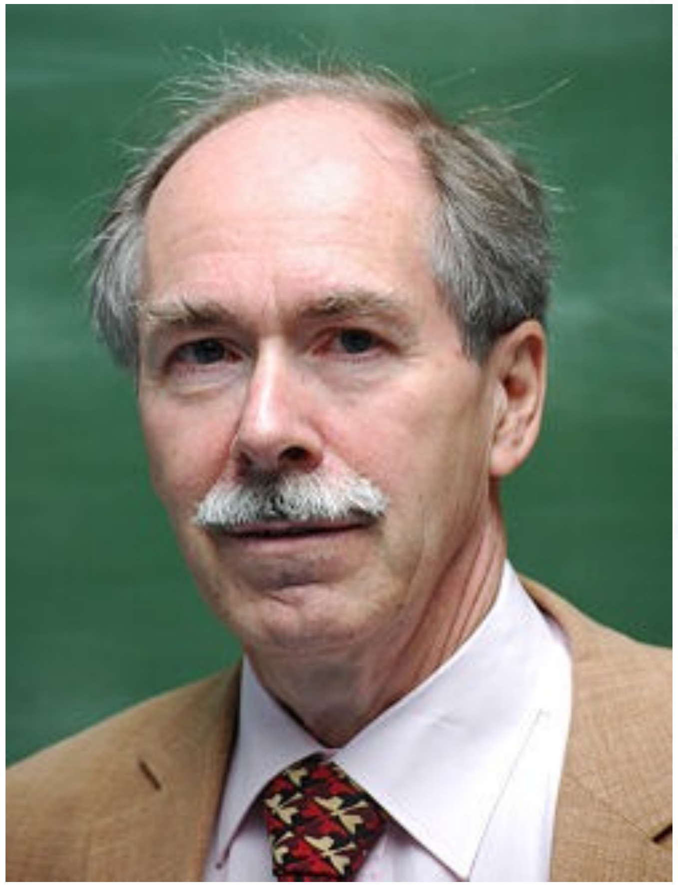


Reviews: Di Piazza et al, 2012; US National Academies, 2018

Resurgence in Asymptotically Free Quantum Field Theory

CAN WE MAKE SENSE OUT OF "QUANTUM CHROMODYNAMICS"?

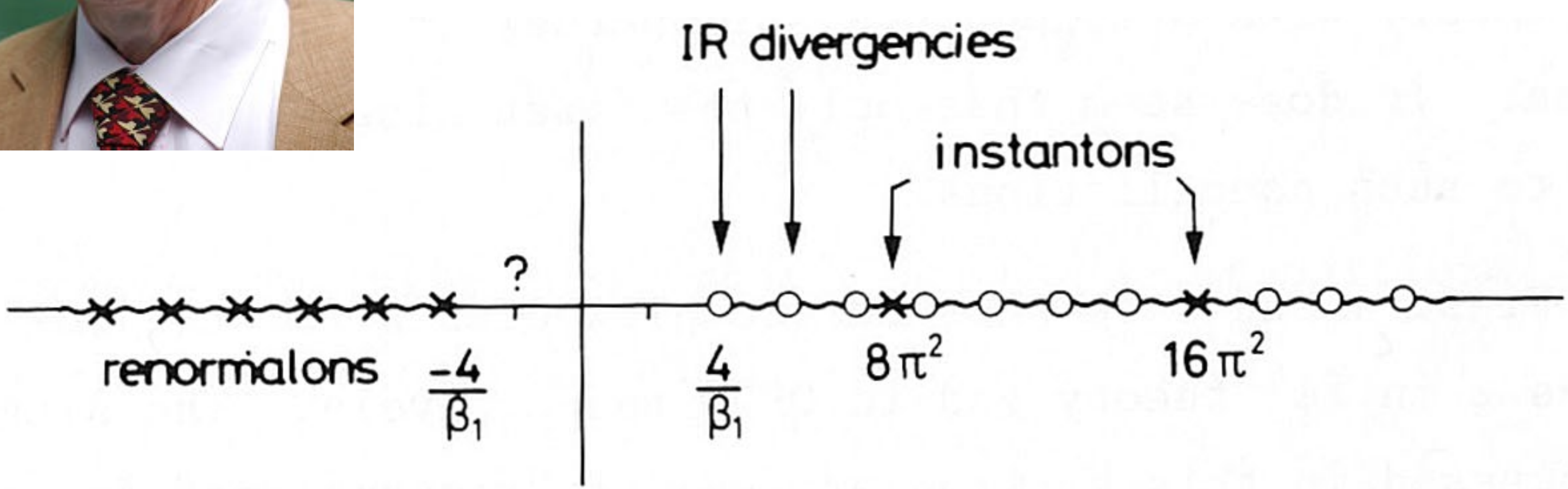
1979



G. 't Hooft

Institute for Theoretical Physics

University of Utrecht, Netherlands



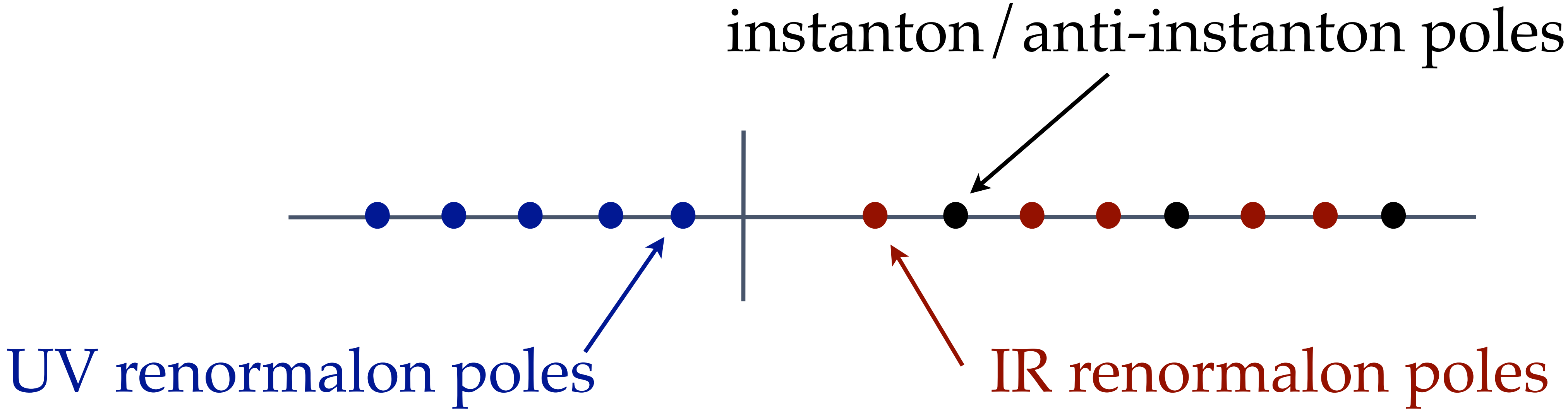
"infrared renormalon puzzle": the BZJ cancelation appears to fail …

Resurgence in Quantum Field Theory

infrared renormalon puzzle of asymptotically free QFT

perturbation theory + Borel: $\longrightarrow$ $+i\,\exp\left[-\frac{2\,S_I}{g^2\,\beta_1}\right]$

non-perturbative instantons : $\longrightarrow$ $-i\,\exp\left[-\frac{2\,S_I}{g^2}\right]$



Resurgence in Quantum Field Theory

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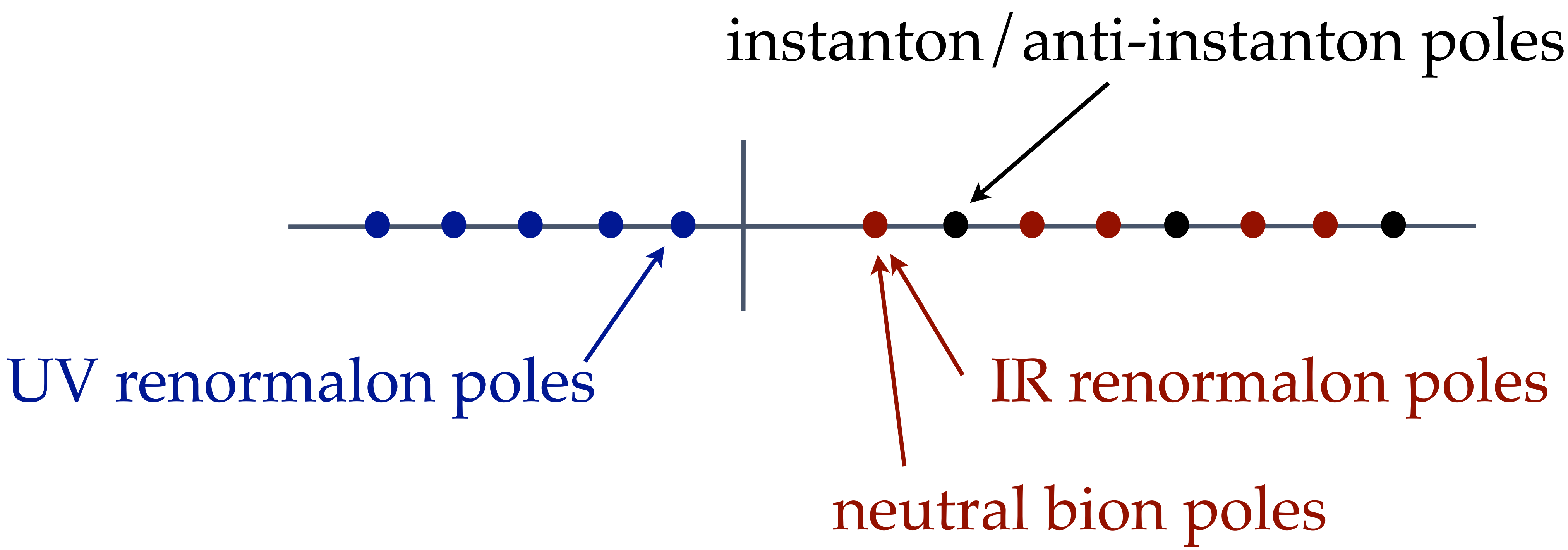


$$\longrightarrow \quad -i\,\exp\left[-\frac{2\,S_I}{g^2\,\beta_1}\right]$$

new non-perturbative objects ("neutral bions") lead to
Bogomolny/Zinn-Justin style resurgent cancelation

GD/ Ünsal, 2012

Resurgence in Quantum Field Theory: recent progress

- 2d sigma models: CP^{N-1}, principal chiral model, O(N), …
- matrix models: unitary, quartic, cubic, Chern-Simons, …
- "localizable" QFT : matrix models
- SUSY QFT and integrable models
- topological string theories
- numerical gradient flow: Thirring model, Bose gas, non-equil., …
- "large N": new large N instantons & non-perturbative completions
- …

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) = \sum_{\text{thimble}} \mathcal{N}_{\text{th}}\, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \, \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

Lefschetz thimble = "functional steepest descents contour"

on a thimble, the path integral becomes
well-defined and computable !

complexified gradient flow:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

CRISTOFORETTI *et al.* (2013)

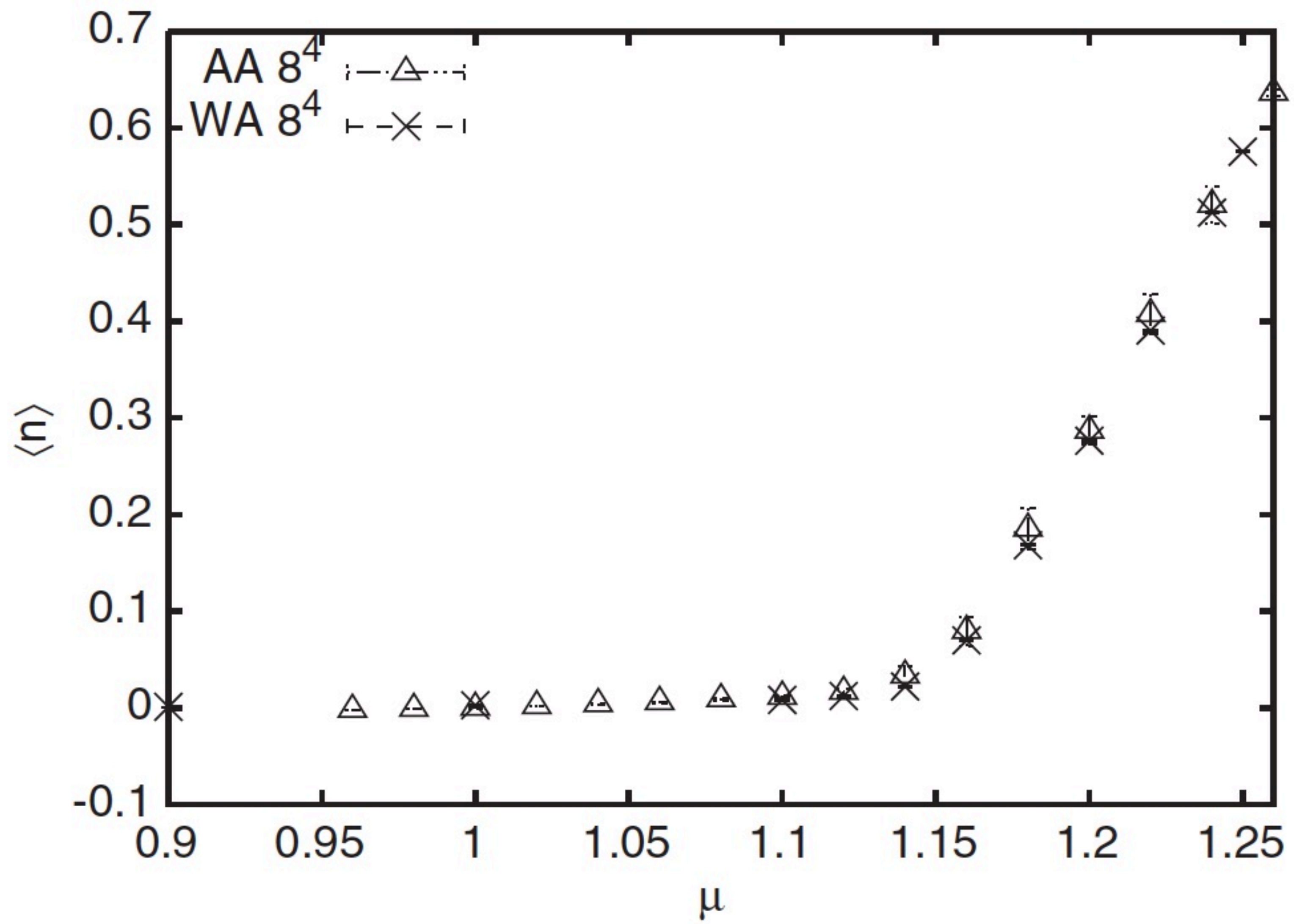


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA)

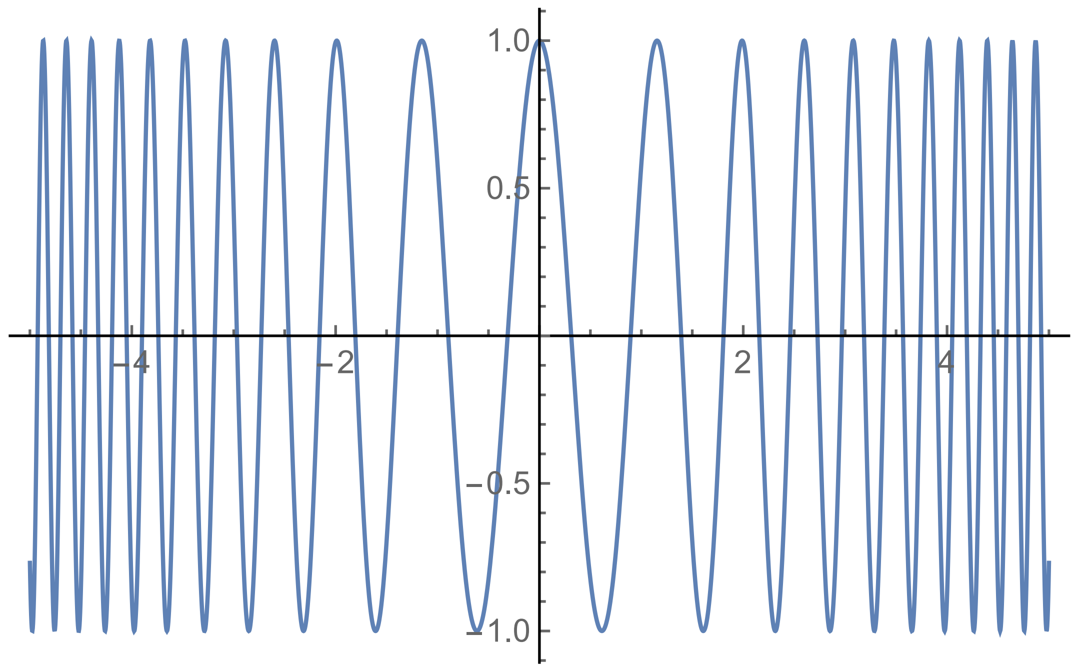
- 4d relativistic Bose gas: complex scalar field theory
- Monte Carlo on thimble softens the sign problem
- results comparable to "worm algorithm"

Generalized Thimble Method

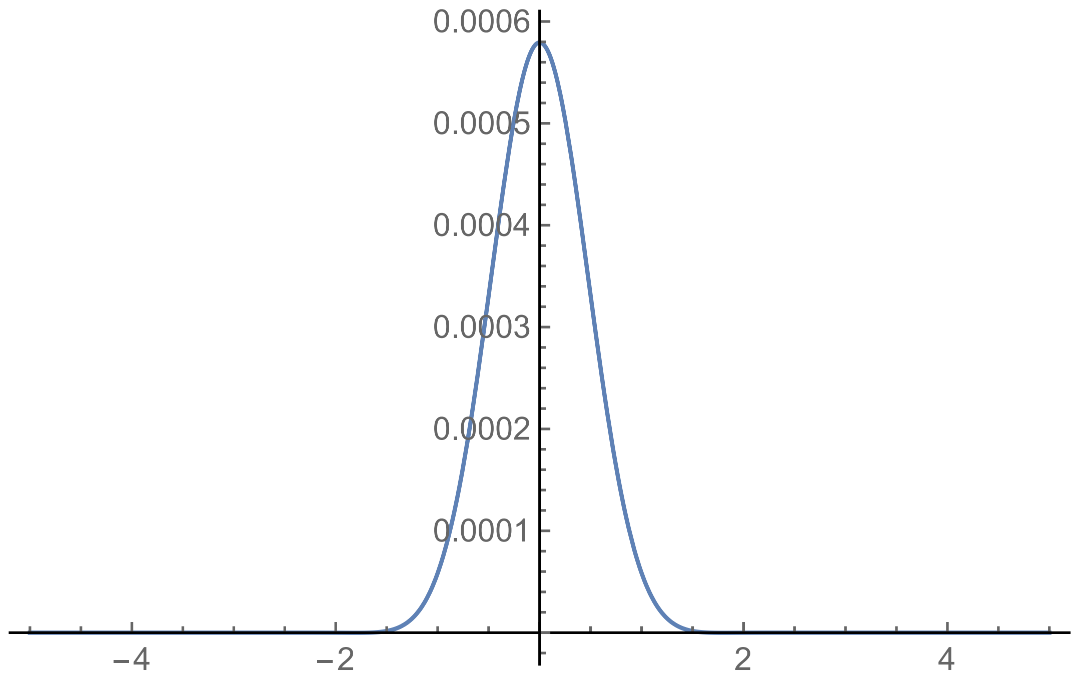
Alexandru, Basar, Bedaque et al 2016

idea: flow to an approximate Lefschetz thimble

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}x^3 + z\,x\right)}dx$$



exact steepest descents contour



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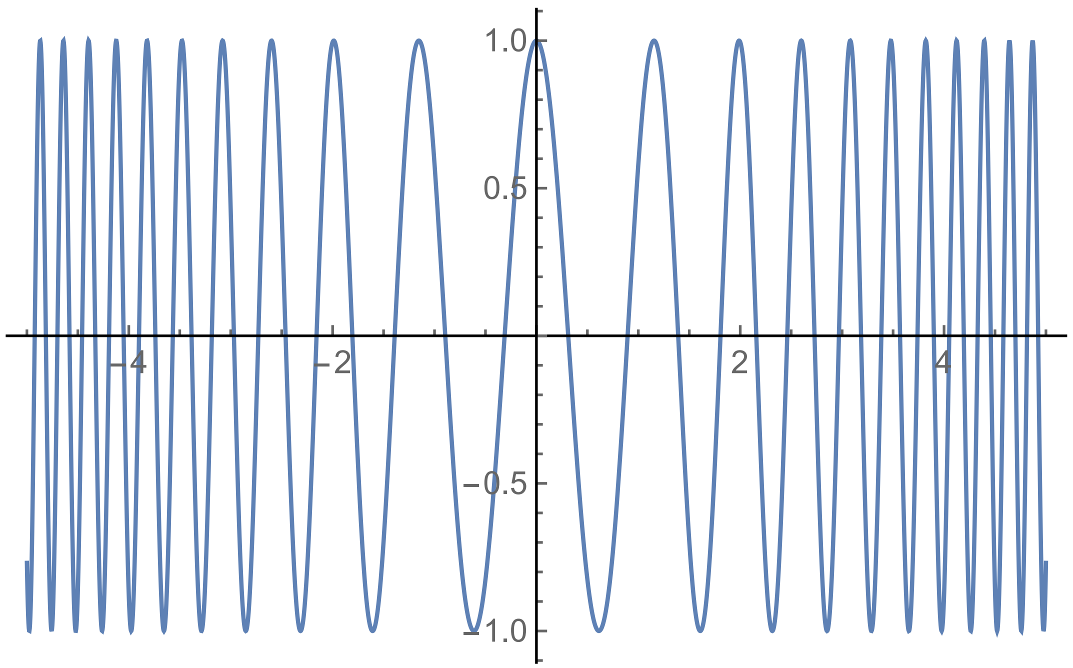

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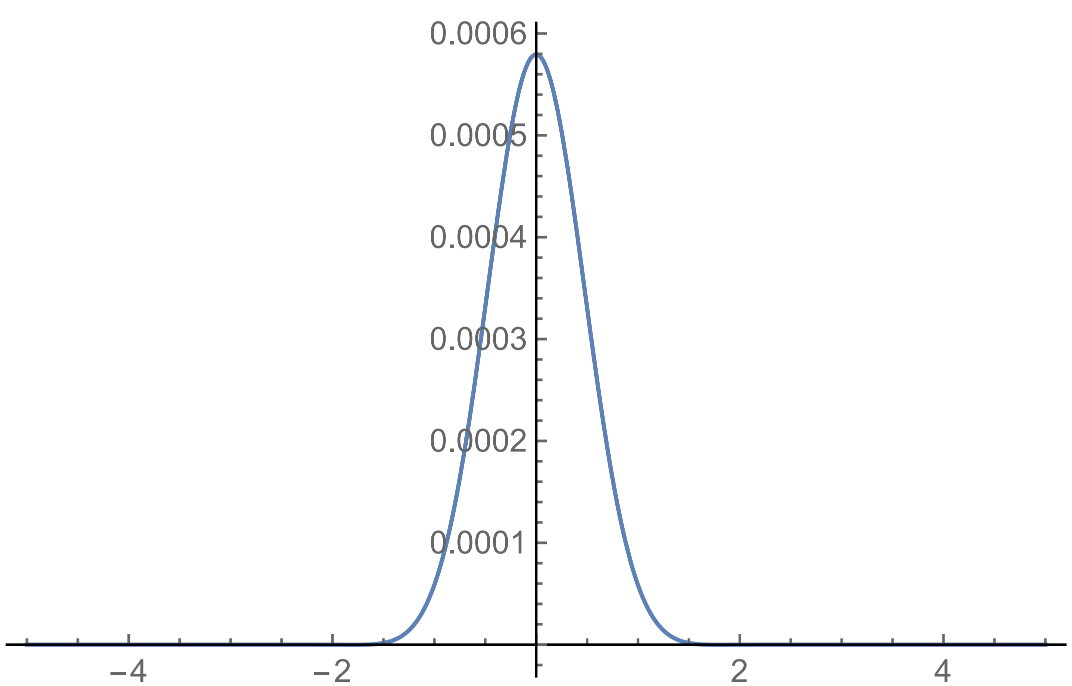
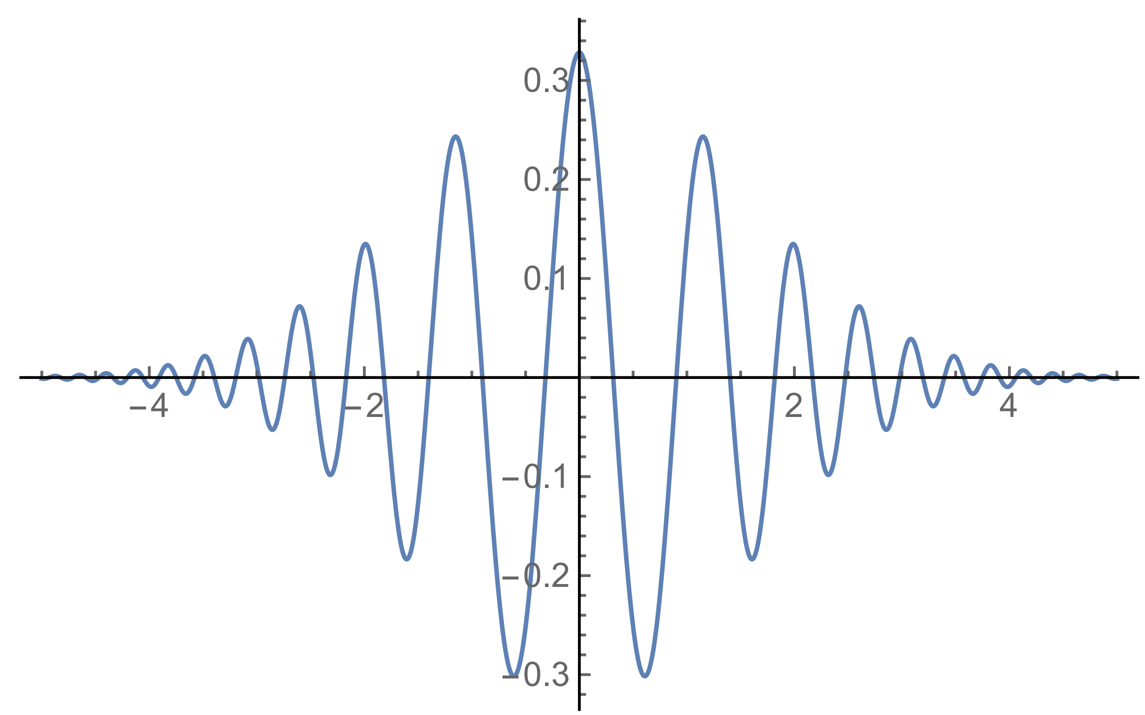


approximate steepest descents contour



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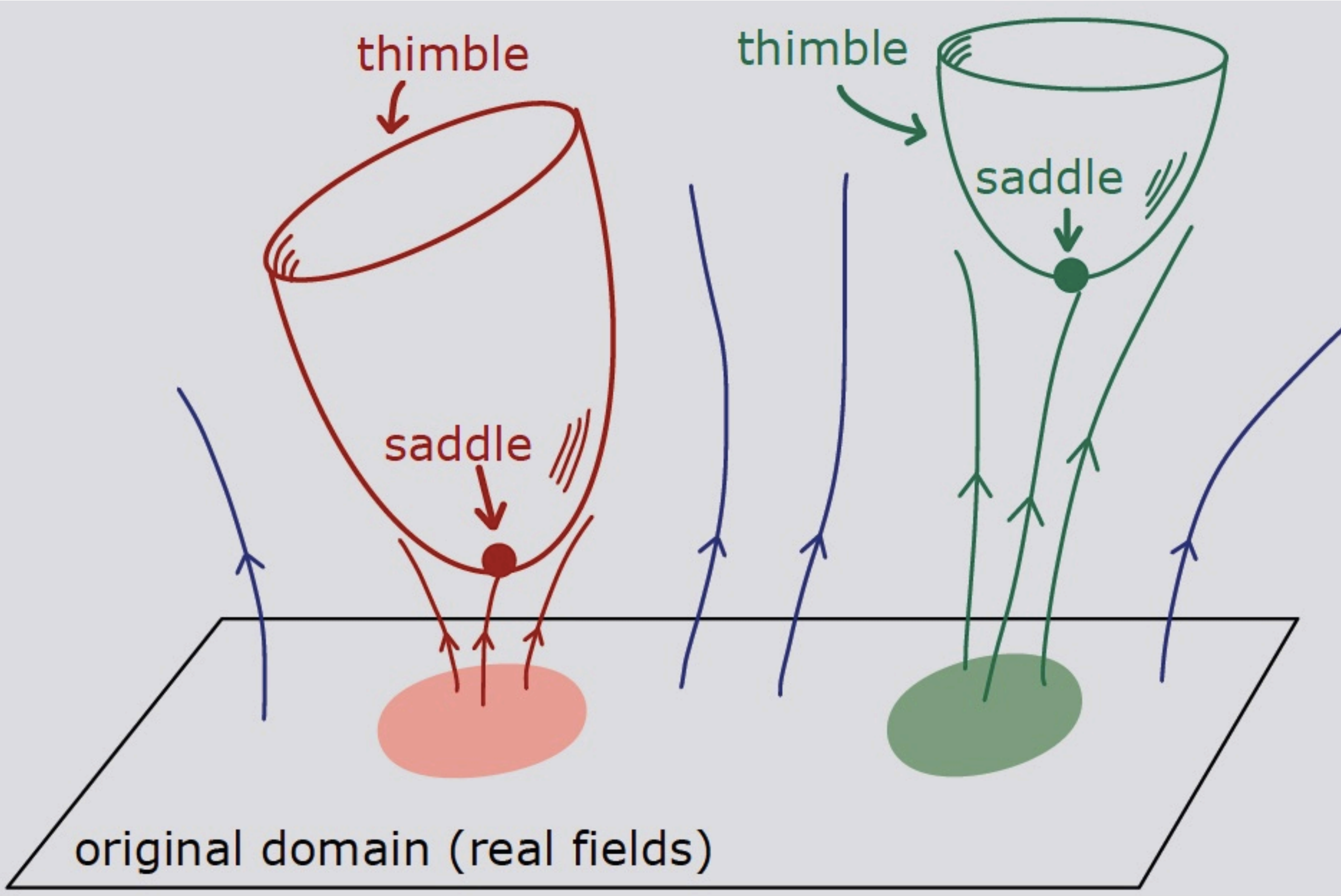


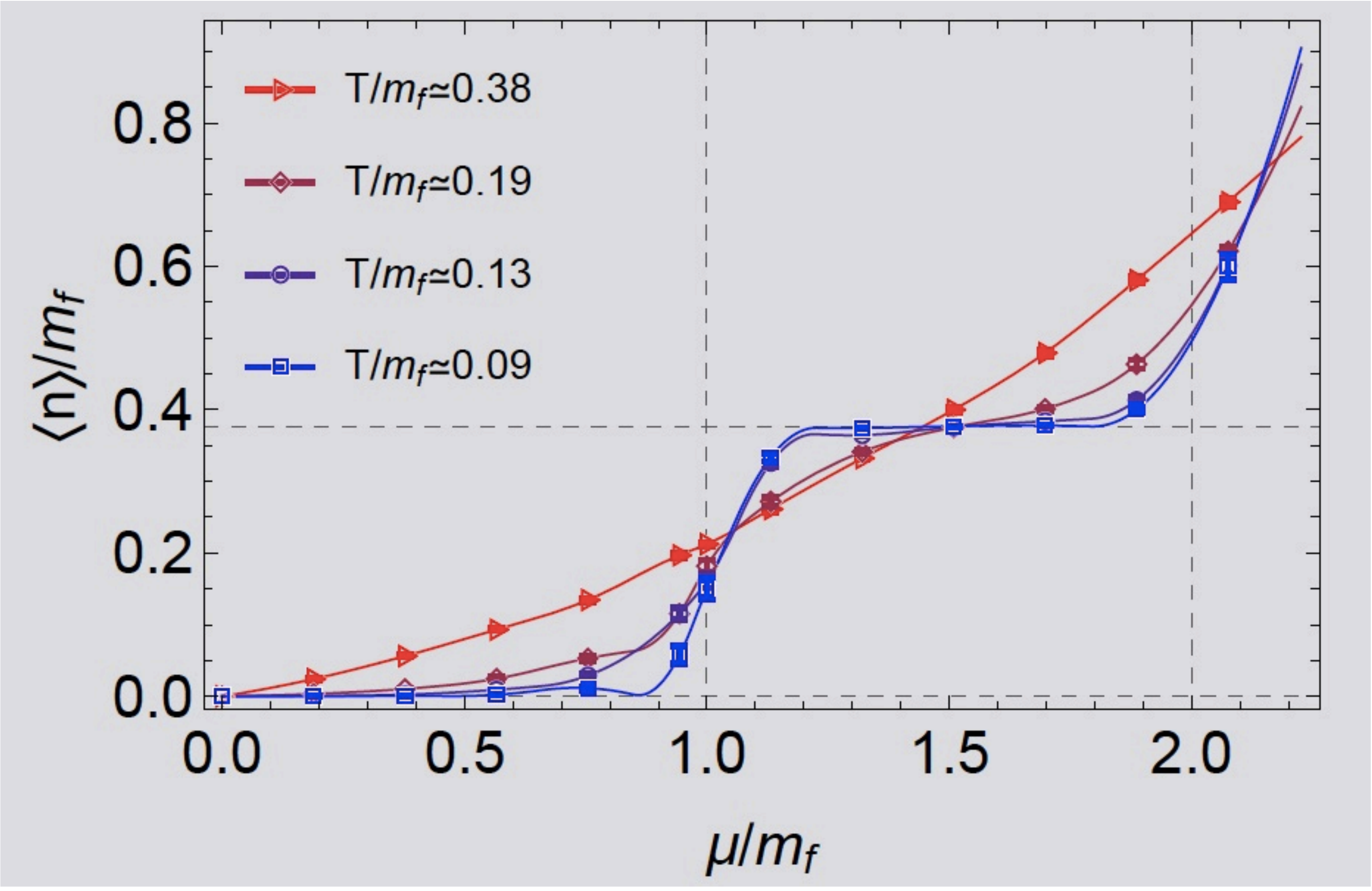
Phase Transitions in QFT: 2d Thirring Model

$$\mathcal{L} = \bar{\psi}^a \left(\gamma_\nu \partial_\nu + m + \mu\, \gamma_0\right) \psi^a + \frac{g^2}{2N_f} \left(\bar{\psi}^a \gamma_\nu \psi^a\right) \left(\bar{\psi}^b \gamma_\nu \psi^b\right)$$

- chain of interacting fermions: asymptotically free
- prototype for dense quark matter
- sign problem at nonzero density
- 2d cousin of Hubbard model

Monte Carlo thimble computation

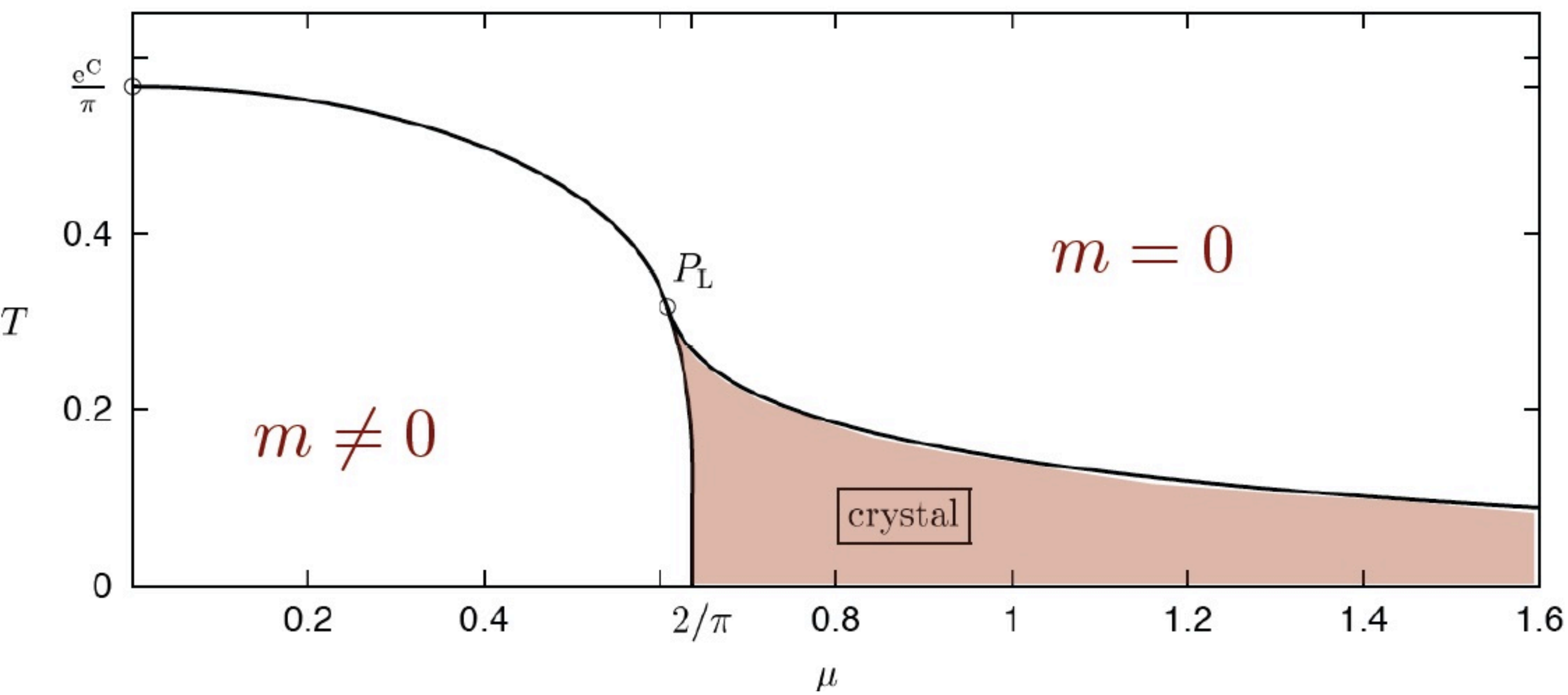
(Alexandru et al, 2016)



Phase Transitions in Gross-Neveu Model

$$\mathcal{L}_{\text{Gross-Neveu}} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g^2}{2}\left(\bar{\psi}_a \psi_a\right)^2$$

- asymptotically free; dynamical mass; chiral symmetry; model for QCD
- large N_f chiral symmetry breaking phase transition
- physics = (relativistic) Peierls dimerization instability in 1+1 dim.



chiral symmetry breaking condensate

$$\sigma(x; T, \mu) \equiv \langle \bar{\psi}\psi \rangle (x; T, \mu)$$

develops crystalline phases !

(Thies et al)

saddles solve inhomogeneous gap equation

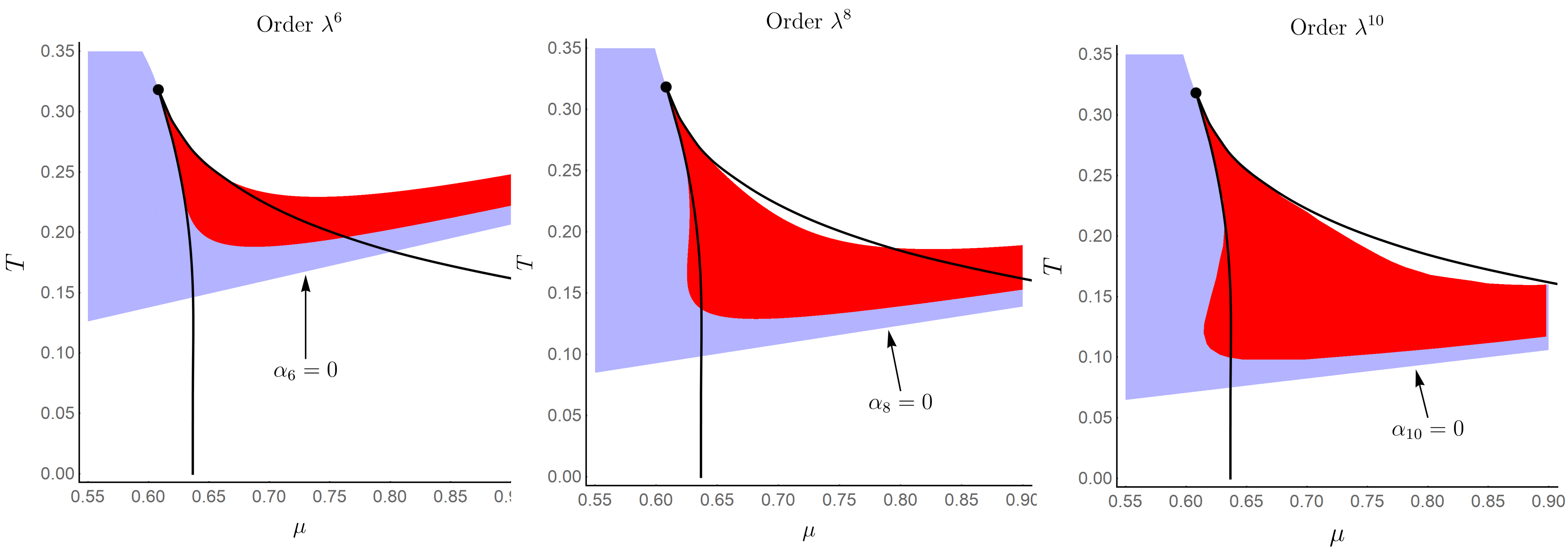
$$\sigma(x; T, \mu) = \frac{\delta}{\delta\sigma(x; T, \mu)} \ln\det\left(i\,\partial\!\!\!/ - \sigma(x; T, \mu)\right)$$

Phase Transitions in Gross-Neveu Model

- thermodynamic potential

$$\Psi[\sigma; T, \mu] = -T \int dE\, \rho(E)\, \ln\left(1 + e^{-(E-\mu)/T}\right) = \sum_n \alpha_n(T,\mu) f_n[\sigma(x; T, \mu)]$$

- (divergent) Ginzburg-Landau expansion = mKdV
- exact saddles are known
- successive orders of GL expansion “reveal” crystal phase



- all orders gives full crystal phase … but T=0 critical point is difficult

Phase Transitions in Gross-Neveu Model

- density expansion has non-perturbative terms: “trans-series”

- high-density expansion at T=0: (convergent)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\rho^2\left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots\right)$$

- low-density expansion at T=0: (non-perturbative trans-series)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{-k/\rho}}{\rho^{k-2}}\mathcal{F}_{k-1}(\rho)$$

- T=0 quantum phase transition

$$\mu_{\rm critical} = \frac{2}{\pi} \quad \leftrightarrow \quad \rho = 0$$

Resurgence in Matrix Models at Large N

3rd order phase transition in Gross-Witten-Wadia unitary matrix model

$$Z(\lambda, N) = \int_{U(N)} DU \, \exp\left[\frac{N}{\lambda} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

Z depends on <u>two parameters</u>

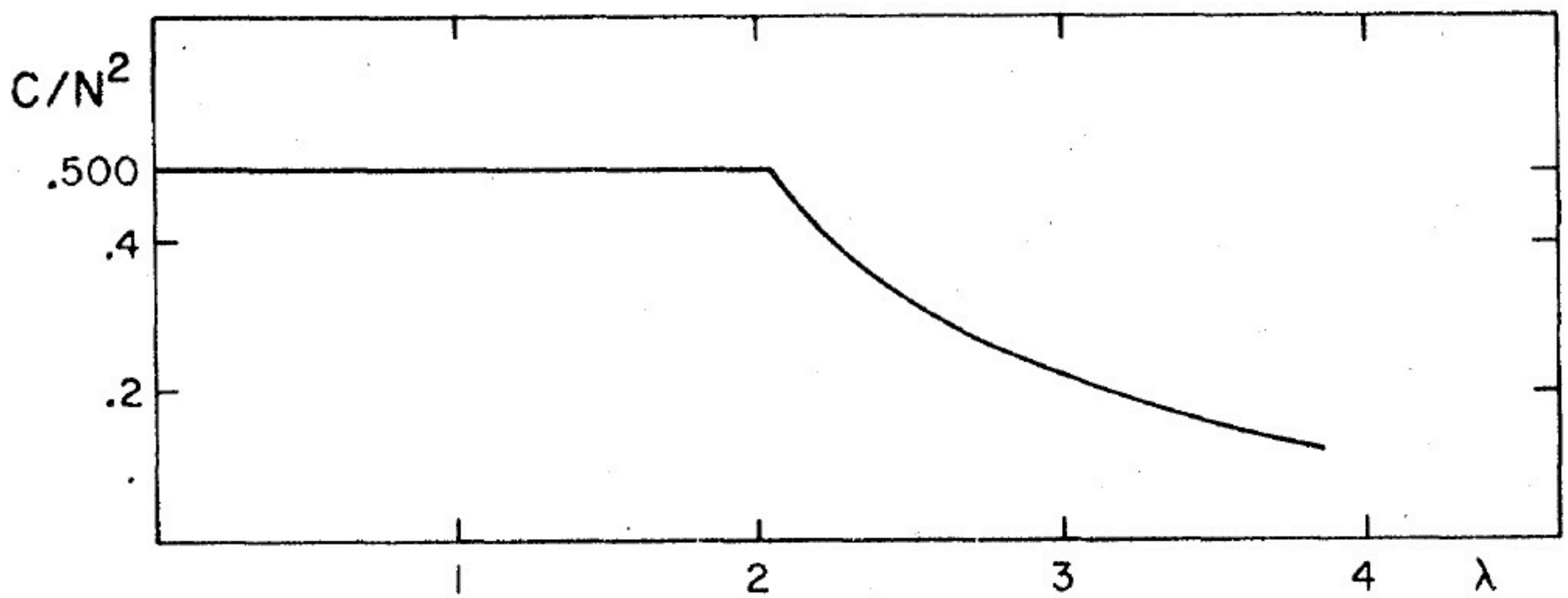


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

"order parameter" $\Delta(t, N) \equiv \langle \det U \rangle$ satisfies a Painleve equation

Resurgence in Matrix Models at Large N

large N weak coupling trans-series:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i\,\sigma_{\rm weak}}{2\sqrt{2\pi N}}\frac{t\,e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

weak coupling large N action:

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\arctan\left(\sqrt{1-t}\right)$$

“one-instanton" fluctuations:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

Resurgence in Matrix Models at Large N

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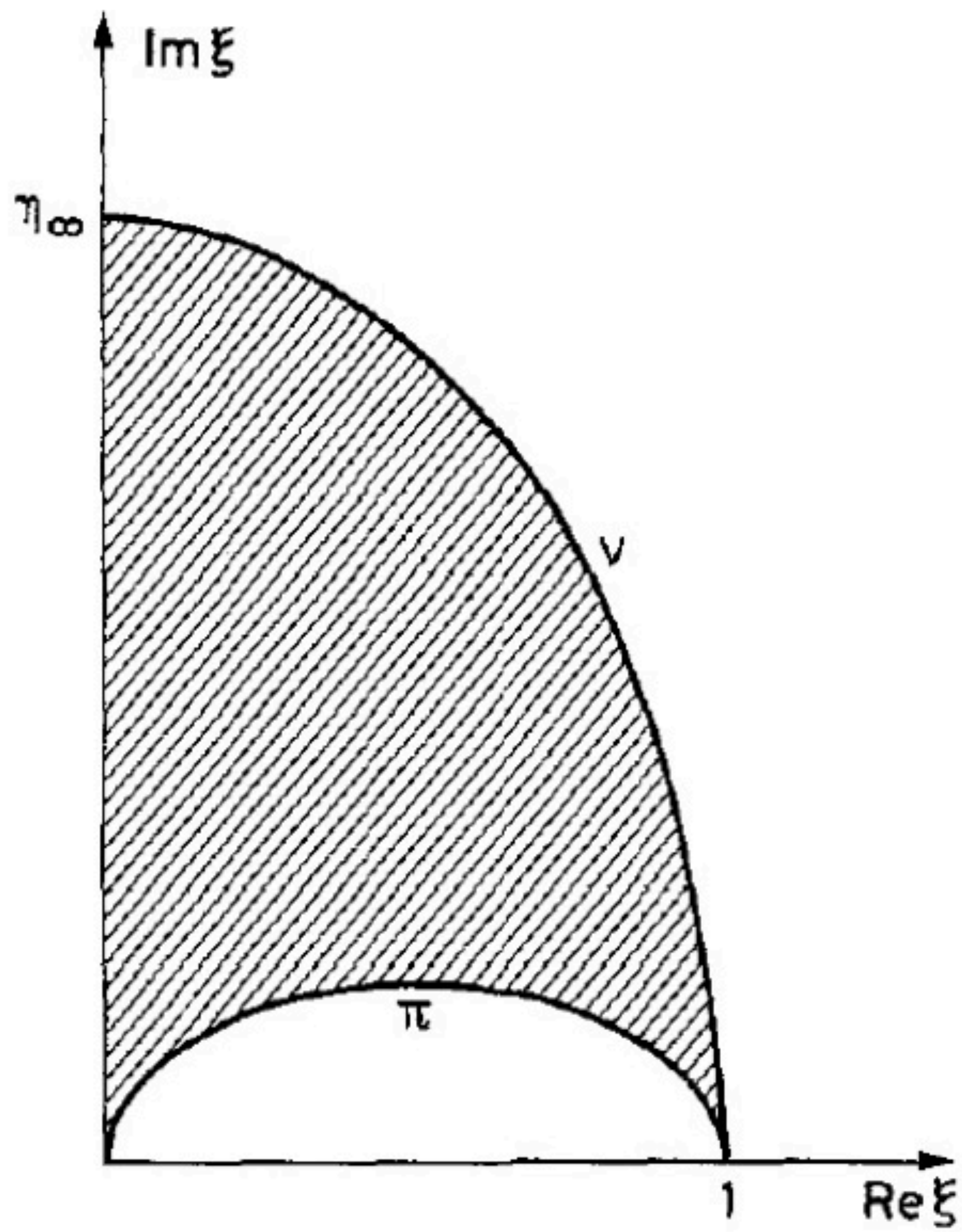
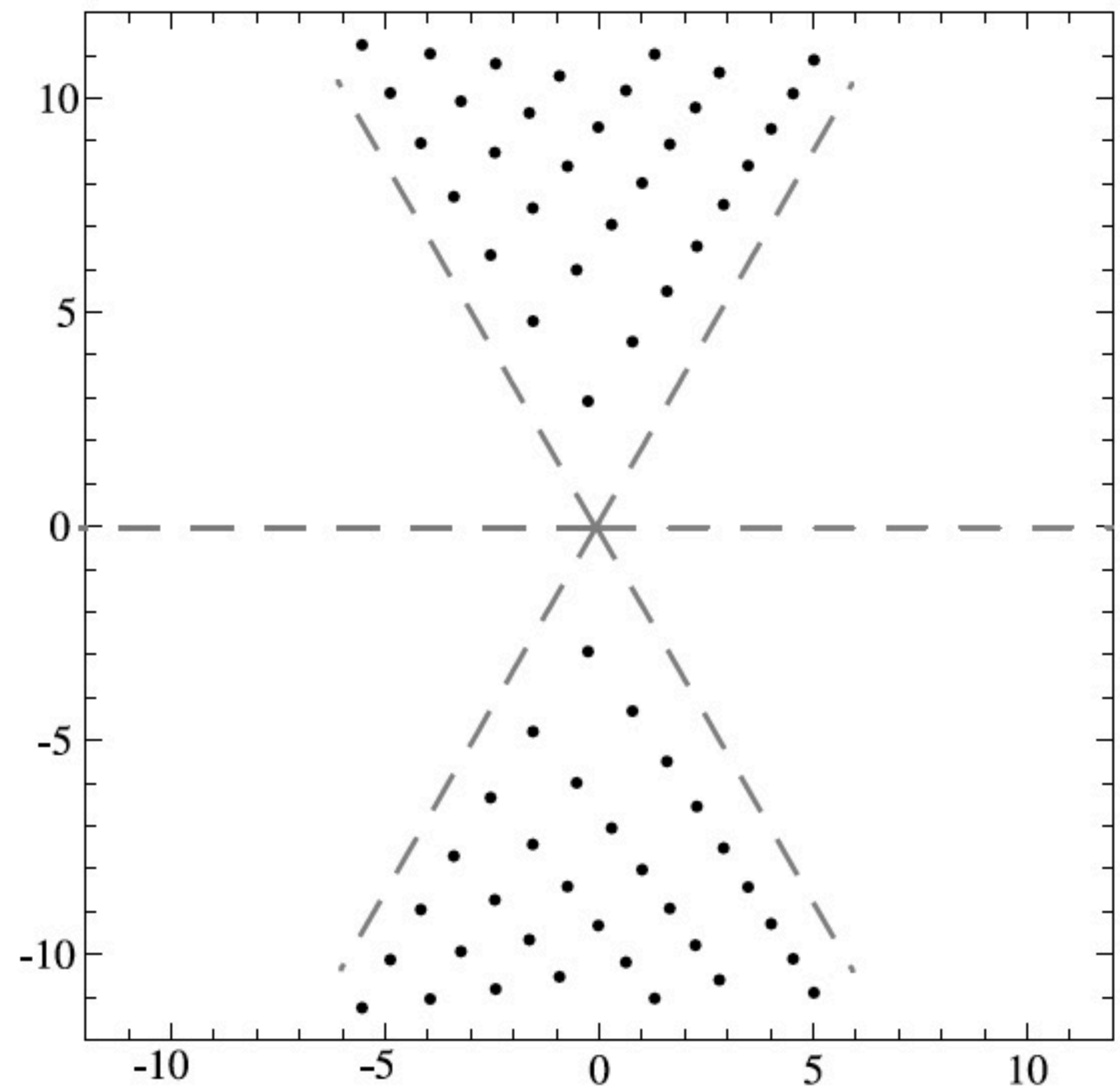
resurgence: large-order growth of “perturbative coefficients”:

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right]$$

Resurgence and Large N Phase Transitions in Matrix Models

Lee-Yang: complex zeros of Z pinch the real axis at the phase transition, in the thermodynamic (large N) limit





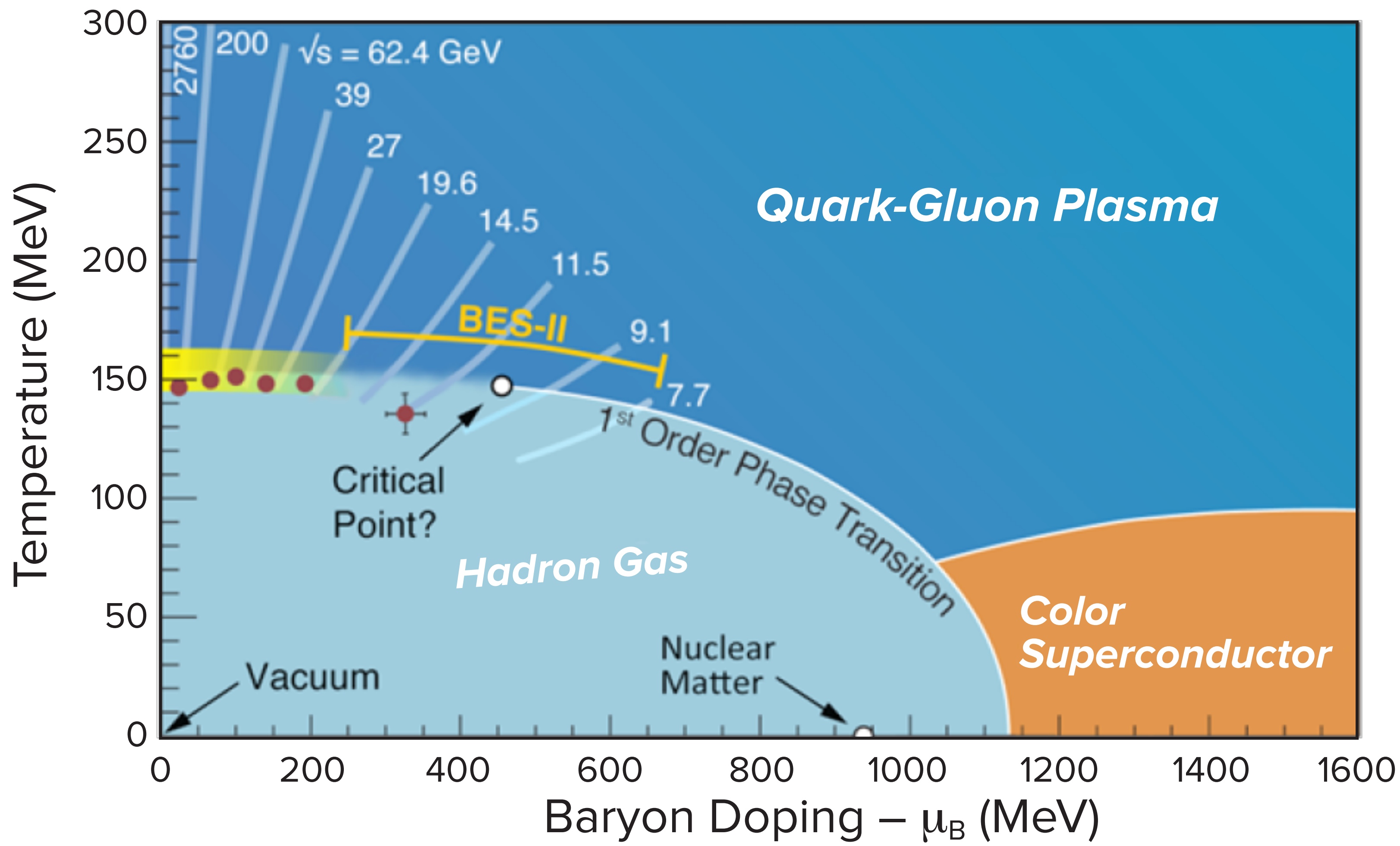
complex parameters can indicate phase transitions

Resurgent Extrapolation

- sometimes asymptotics is the ONLY thing we can do

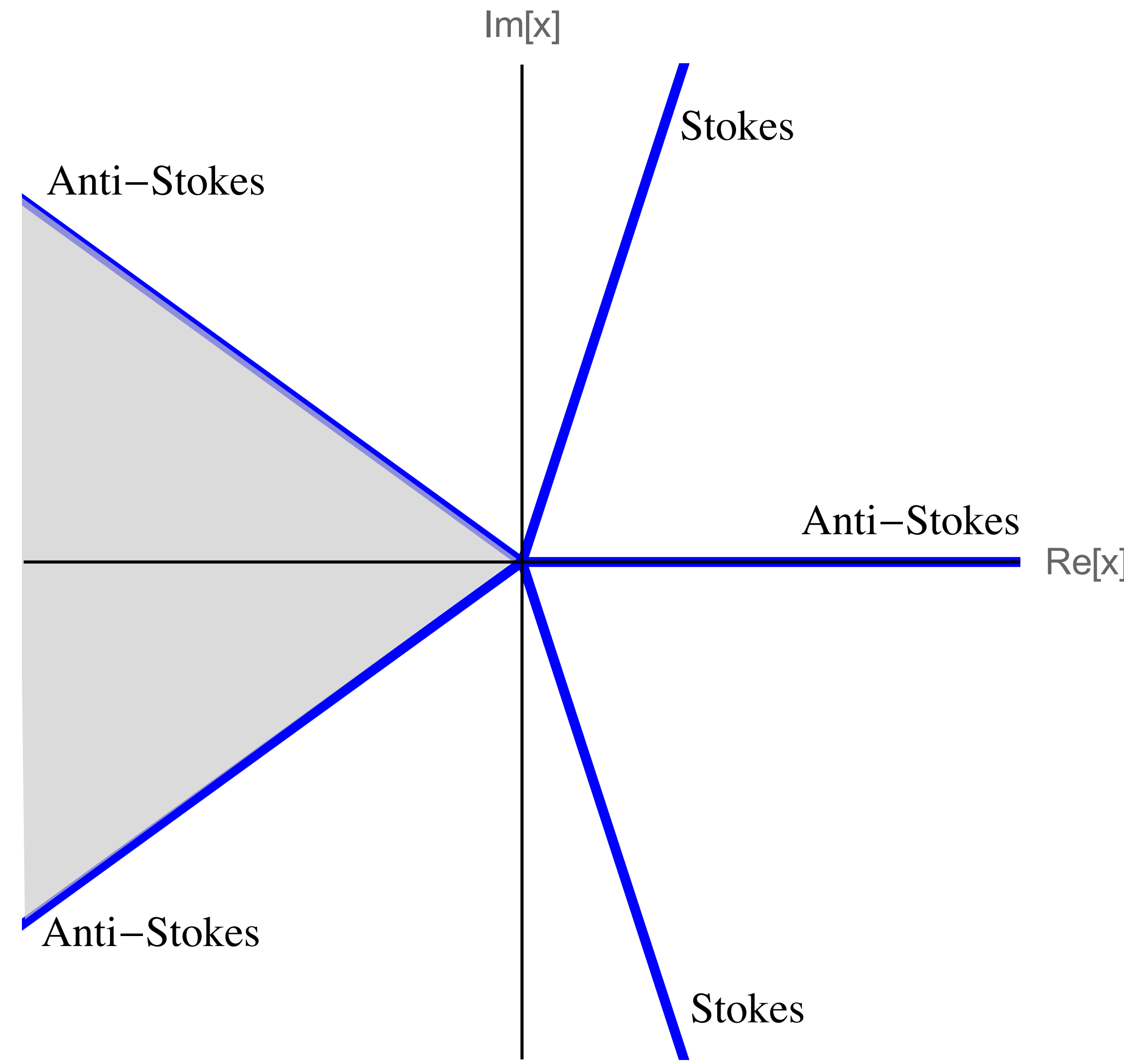
- question: how much global information can be decoded from a FINITE number of perturbative coefficients ?

- how much information is required to <u>see</u> and to <u>probe</u> a phase transition ?



Resurgent Extrapolation

- case-study: Painleve I equation $$y''(x) = 6\,y^2(x) - x$$

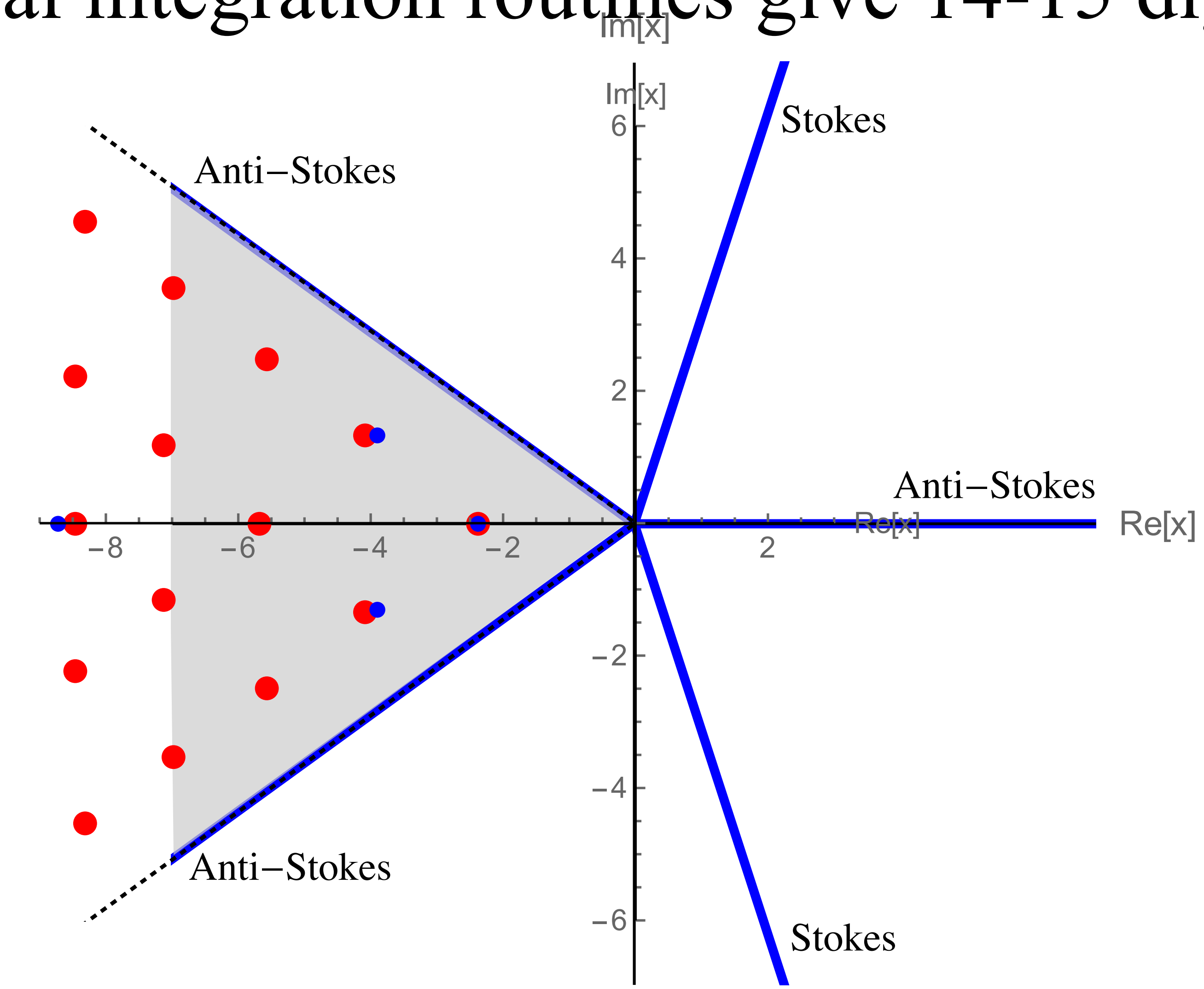


- Painleve I equation has 5 sectors in the complex x plane, separated by phase transitions

- suppose we expand about x=+infty to finite order N: how much do these coefficients “know” about the other sectors?

Resurgent Extrapolation

- 10 terms at +infty encode 23 digits precision at x=0
- best numerical integration routines give 14-15 digits precision at x=0



- we can also explore the complex plane: with just N=10 terms, we can “see” the first 3 poles; with N=50 terms, we see the first 15 poles

- resurgent extrapolation can decode global behavior from surprisingly little input data from some other regime

Resurgent Extrapolation

- resurgent extrapolation can decode global behavior from surprisingly little input data from some other regime
- transmutation of the trans-series:
- near $x \to +\infty$

$$y(x) \sim -\sqrt{\frac{x}{6}} \sum_{n=0}^{\infty} a_n \frac{1}{x^{5n/2}} \quad , \qquad x \to +\infty$$

- in the pole region $\frac{4\pi}{5} \leq \arg(x) \leq \frac{6\pi}{5}$

$$y(x) \approx \frac{1}{(x - x_{\text{pole}})^2} + \frac{x_{\text{pole}}}{10}(x - x_{\text{pole}})^2 + \frac{1}{6}(x - x_{\text{pole}})^3$$
$$+h_{\text{pole}}(x - x_{\text{pole}})^4 + \frac{x_{\text{pole}}^2}{300}(x - x_{\text{pole}})^6 + \dots$$

- this phase transition is encoded in fluctuations at x=+infty

Conclusions

- "resurgence" is a new and improved form of asymptotics
- deep connections between perturbative and non-perturbative physics
- recent applications to differential eqs, QM, QFT, string theory, …
- outlook: new theoretical approach to quantum systems in extreme conditions
- outlook: computational definition of real-time path integrals
- outlook: computational access to strongly-coupled systems, phase transitions, particle production, and far-from-equilibrium physics
- outlook: new insights to hydrodynamics